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The main purpose of this work is to identify invariant quadratic operators associated with Linear Canonical Transformations (LCTs) which could play important roles in physics. In quantum physics, LCTs are the linear transformations which keep invariant the Canonical Commutation Relations (CCRs). In this work, LCTs corresponding to a general pseudo-Euclidian space are considered and related to a phase space representation of quantum theory. Explicit calculations are firstly performed for the monodimensional case to identify the corresponding LCT-invariant quadratic operators then multidimensional generalizations of the obtained results are deduced. The eigenstates of these operators are also identified. A first kind of LCT-invariant operator is a second order polynomial of the coordinates and momenta operators. The coefficients of this polynomial depend on the mean values and the statistical variances-covariances of the coordinates and momenta operators themselves. It is shown that these statistical variances-covariances can be related with thermodynamic variables. In this context, new quantum corrections to the ideal gas state equation are deduced from correction to the Hamiltonian operator of non-relativistic free quantum particles that is suggested by LCT-covariance. Two other LCT-invariant quadratic operators, which can be considered as the number operators of some quasiparticles, are also identified: the first one is a number operator of bosonic type quasiparticles and the second one corresponds to fermionic type. This fermionic LCT-invariant quadratic operator is directly related to a spin representation of LCTs. It is shown explicitly, in the case of a relativistic pentadimensional theory, that the eigenstates of this operator can be considered as basic quantum states of elementary fermions. A classification of the fundamental fermions, compatible with the Standard Model of particle physics, is established from a classification of these states.

1. Introduction

Linear Canonical Transformations (LCTs) can be considered as generalization of some useful integral transformations like Fourier and Fractional Fourier Transforms. They are studied and used in many areas [1–12]. In quantum theory, they can be identified as the linear transformations which keep invariant the Canonical Commutation Relations (CCRs) defining the coordinates and momenta operators. In the references [11, 12], it was shown that a covariance principle related with LCTs may play an important role in physics and in the establishment of a unified theory of fundamental interactions. In the present work, our main objective is to

identify some quadratic operators that are invariant under the action of LCTs and to highlight their potential importance in physics.

Some operators related to LCTs and their representations were already considered by various authors [5–7]. However, the quadratic operators that are identified in this work are new ones even if some of them can be considered as generalization of operators introduced in the references [10–12].

In the next section (section 2) we consider a study about a relation that exists between the theory of simple Linear Harmonic oscillator and monodimensional LCTs. This study leads us also to the introduction of a correction to the Hamiltonian of a free particle in non-relativistic quantum physics. The new Hamiltonian thus obtained is related to an LCT-invariant quadratic operator and its introduction brings new solutions to some inconveniences related to the older Hamiltonian. This section 2 highlights some interesting basic facts and results that are extended and generalized in the identification and study of the invariant quadratic operators associated to general multidimensional LCTs. These extensions and generalizations are performed in the sections 3 and 5. The statistical variance-covariances associated to the coordinate and momenta operators have an important place in the formalism that is established. In the section 4, it is shown that these statistical parameters can be practically related to thermodynamic variables. In this framework, new quantum corrections to the ideal gas equation are deduced from the correction that has been considered to the Hamiltonian operator of free particles. The main results obtained through this work are listed in the section 6 and conclusions are given in the section 7. Some of the possible applications of these results are also discussed in these sections. The notation system which is considered is inspired from the reference [13]. Boldfaced letter like \mathbf{p} are mainly used for quantum operator.

2. Quantum linear harmonic oscillator and particular LCTs

2.1. Hamiltonian as invariant quadratic operator associated to particular LCTs

An example of a well-known quadratic operator that can be considered as invariant under the action of some particular monodimensional LCTs, in the framework of non-relativistic quantum theory, is the Hamiltonian operator \mathbf{H} of a monodimensional simple harmonic oscillator. For a quantum harmonic oscillator [14, 15] with a mass m , angular frequency ω , momentum operator \mathbf{p} and coordinate operator \mathbf{x} , the expression of this Hamiltonian operator is

$$\mathbf{H} = \frac{(\mathbf{p})^2}{2m} + \frac{1}{2}m\omega^2(\mathbf{x})^2 \quad (1)$$

We may consider the set of linear transformations of the form

$$\begin{cases} \mathbf{p}' = \mathbf{a}\mathbf{p} + \mathbf{b}\mathbf{x} \\ \mathbf{x}' = \mathbf{c}\mathbf{p} + \mathbf{d}\mathbf{x} \end{cases} \quad \text{with} \quad \begin{pmatrix} \mathbf{a} & \mathbf{c} \\ \mathbf{b} & \mathbf{d} \end{pmatrix} = \begin{pmatrix} \cos \theta & -\frac{\sin \theta}{m\omega} \\ m\omega \sin \theta & \cos \theta \end{pmatrix} \quad (2)$$

The relations (2) correspond to a LCT i.e. linear transformation which leaves invariant the CCR. We have the relations (\hbar is the reduced Planck constant)

$$\mathbf{a}\mathbf{d} - \mathbf{b}\mathbf{c} = 1 \Leftrightarrow [\mathbf{p}', \mathbf{x}'] = [\mathbf{p}, \mathbf{x}] = -i\hbar \quad (3)$$

The Hamiltonian operator \mathbf{H} in (1) is invariant under the action of the particular LCTs in (2)

$$\mathbf{H}' = \frac{(\mathbf{p}')^2}{2m} + \frac{1}{2}m\omega^2(\mathbf{x}')^2 = \frac{(\mathbf{p})^2}{2m} + \frac{1}{2}m\omega^2(\mathbf{x})^2 = \mathbf{H} \quad (4)$$

It will be noticed through the next sections that the expression of some general invariant quadratic operators associated to LCTs has a similarity with this Hamiltonian operator. Because of this similarity, the formalism that is considered in the identification of the eigenstates of these operators shares analogy to the well-known formalism associated with the theory of harmonic oscillators. These eigenstates present some similarities with what are called coherent states, generalized coherent states and squeezed states in the literature [16–19]. It follows that the formalism considered in the present work can also be considered as an extension and generalization of the theory of quantum harmonic oscillators with an establishment of a link between it and a general theory of linear canonical transformations, phase space representation of quantum theory, thermodynamic and particle physics.

2.2. Statistical parameters, dispersion operators and ladder operators

It is well known that the eigenvalues equations of the Hamiltonian H in (1) is of the form

$$H|n\rangle = \left(n + \frac{1}{2}\right)\hbar\omega|n\rangle \tag{5}$$

with n a non-negative integer. The wavefunctions corresponding to a state $|n\rangle$ in coordinate representations is of the form

$$\langle x|n\rangle = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) e^{-\frac{m\omega}{2\hbar}x^2} \tag{6}$$

With H_n a Hermite polynomial of degree n . It can be easily deduced that the statistical mean values $\langle p \rangle$ and $\langle x \rangle$ of the momentum and coordinate operators associated to any eigenstate $|n\rangle$ of the Hamiltonian and the statistical variance \mathcal{B} and \mathcal{A} of these operators associated to the ground state $|0\rangle$ are respectively

$$\begin{cases} \langle p \rangle = \langle n|p|n \rangle = 0 & \langle x \rangle = \langle n|x|n \rangle = 0 \\ \mathcal{B} = \langle 0|(p - \langle p \rangle)^2|0 \rangle = \frac{m\hbar\omega}{2} = \frac{\hbar^2}{4\mathcal{A}} \\ \mathcal{A} = \langle 0|(x - \langle x \rangle)^2|0 \rangle = \frac{\hbar}{2m\omega} = \frac{\hbar^2}{4\mathcal{B}} \end{cases} \tag{7}$$

Using these statistical parameters, the expression of the Hamiltonian (4) can be written in the form

$$H = \frac{(p)^2}{2m} + \frac{1}{2}m\omega^2(x)^2 = \frac{1}{2m}[(p - \langle p \rangle)^2 + 4\mathcal{B}(x - \langle x \rangle)^2] = \frac{\mathfrak{D}^+}{m} \tag{8}$$

in which \mathfrak{D}^+ is the momentum dispersion operator considered in the reference [10]. Here we have $\langle p \rangle = 0$ and $\langle x \rangle = 0$ according to (7) but they are introduced explicitly in the relation (8) to highlight the relation between the Hamiltonian operator and the momentum dispersion operator that will be generalized in the next section.

Using again the relations in (7), the wavefunctions in the relation (6) can also be written in the form

$$\langle x|n, \mathcal{B} \rangle = \left(\frac{2\mathcal{B}}{\pi\hbar^2}\right)^{1/4} \frac{H_n\left(\frac{\sqrt{2\mathcal{B}}}{\hbar}x\right)}{\sqrt{2^n n!}} e^{-\frac{\mathcal{B}}{\hbar^2}x^2} = \left(\frac{1}{2\pi\mathcal{A}}\right)^{1/4} \frac{H_n\left(\frac{1}{\sqrt{2\mathcal{A}}}x\right)}{\sqrt{2^n n!}} e^{-\frac{1}{4\mathcal{A}}x^2} \tag{9}$$

and the eigenvalue equation (5) of the Hamiltonian operator itself can be written in the form

$$H|n, \mathcal{B} \rangle = \frac{\mathfrak{D}^+}{m}|n, \mathcal{B} \rangle = (2n + 1)\frac{\mathcal{B}}{m}|n, \mathcal{B} \rangle \tag{10}$$

The change of notation from $|n\rangle$ to $|n, \mathcal{B} \rangle$ is introduced to highlight the fact that in reality there is also a dependence on \mathcal{B} .

Dirac, in his theory of harmonic oscillator has introduced the ladder operators (annihilation and creation operators). Using the statistical parameters considered previously, the expression of these ladder operators (denoted \mathfrak{z} and \mathfrak{z}^\dagger) can be written in the forms

$$\begin{cases} \mathfrak{z} = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{i}{m\omega}p\right) = \frac{i}{2\sqrt{\mathcal{B}}}\left(p - \frac{2i}{\hbar}\mathcal{B}x\right) = \frac{iz}{2\sqrt{\mathcal{B}}} \\ \mathfrak{z}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{i}{m\omega}p\right) = \frac{-i}{2\sqrt{\mathcal{B}}}\left(p + \frac{2i}{\hbar}\mathcal{B}x\right) = \frac{-iz^\dagger}{2\sqrt{\mathcal{B}}} \end{cases} \quad [\mathfrak{z}, \mathfrak{z}^\dagger] = 1 \tag{11}$$

The operators $\mathfrak{z} = p + \frac{2i}{\hbar}\mathcal{B}x$ is introduced in the relation (11) because its generalization plays an important role in the multidimensional generalization and in the construction of the phase space representation that are considered in the next section. This operator is a particular monodimensional case of the operator $z_\mu = p_\mu + \frac{2i}{\hbar}\mathcal{B}_{\mu\nu}x^\nu$ introduced in the reference [10].

In terms of the ladder operators \mathfrak{z} and \mathfrak{z}^\dagger , we have for the hamiltonian H the expression

$$H = \frac{\mathfrak{D}^+}{m} = \frac{1}{4m}(z^\dagger z + z z^\dagger) = \frac{\mathcal{B}}{m}(\mathfrak{z}^\dagger \mathfrak{z} + \mathfrak{z} \mathfrak{z}^\dagger) = \frac{\mathcal{B}}{m}(2\mathfrak{z}^\dagger \mathfrak{z} + 1) \tag{12}$$

The eigenvalue equation (9) can be itself deduced from the relation (12) and the commutation relation in (11). We have also the well-known relations

$$\begin{cases} \mathfrak{z}|n, \mathcal{B}\rangle = \sqrt{n}|n-1, \mathcal{B}\rangle \\ \mathfrak{z}^\dagger|n, \mathcal{B}\rangle = \sqrt{n+1}|n+1, \mathcal{B}\rangle \end{cases} \quad (13)$$

which justify the name ladder operators for \mathfrak{z} and \mathfrak{z}^\dagger . Under the action of the LCT (2), the law of transformations of the ladder operators \mathfrak{z} and \mathfrak{z}^\dagger correspond to unitary transformations

$$\begin{cases} \mathfrak{z}' = e^{i\theta}\mathfrak{z} \\ \mathfrak{z}'^\dagger = e^{-i\theta}\mathfrak{z}^\dagger \end{cases} \quad (14)$$

The unitary transformation (14) is also known as a Bogoliubov transformation. The LCT-invariance of the Hamiltonian can also be seen through the relations (12) and (14).

3. Phase space representation and LCTs: non-relativistic monodimensional case

3.1. Analogy between a non-relativistic free quantum particle and a harmonic oscillator

In the framework of classical mechanics, the simplest phase space is the phase space of a Newtonian material point performing a monodimensional motion. This motion, is classically, completely determined by the instantaneous values of its momentum p and coordinate x . The corresponding phase space is the set $\{(p, x)\}$ of all possible values of (p, x) and the classical state is then completely defined by a point (i.e. an element) in this classical phase space.

In the framework of quantum mechanics, the uncertainty principle tells that it is impossible to have simultaneously exact values of p and x . It follows that the state of a particle cannot be represented with a pint-like couple (p, x) and then the concept of phase space as defined from classical mechanics lead to an ambiguity.

In this section, our goal is to describe a formalism that can lead to an acceptable definition of a quantum phase space. It is achieved through the introduction of a phase space representation of quantum states and operators using some results based on the theory of linear harmonic oscillator and its relation with LCTs as described through the previous section.

In the framework of quantum mechanics, the particle is described by a state $|\psi\rangle$ to which corresponds a coordinate wavefunction $\psi(x)$ and a momentum wavefunction $\tilde{\psi}(p)$. The wavefunctions $\psi(x)$ and $\tilde{\psi}(p)$ are linked by a Fourier transform [14]. Let us denote respectively $\langle p \rangle$, $\langle x \rangle$, \mathcal{B} and \mathcal{A} the mean values and statistical variance of the momentum operator \mathbf{p} and coordinate operator \mathbf{x} of the particle.

$$\begin{cases} \langle \psi | \mathbf{p} | \psi \rangle = \langle p \rangle & \langle \psi | (\mathbf{p} - \langle p \rangle)^2 | \psi \rangle = \mathcal{B} \\ \langle \psi | \mathbf{x} | \psi \rangle = \langle x \rangle & \langle \psi | (\mathbf{x} - \langle x \rangle)^2 | \psi \rangle = \mathcal{A} \end{cases} \quad (15)$$

We have the uncertainty relation

$$\sqrt{\mathcal{A}} \sqrt{\mathcal{B}} \geq \frac{\hbar}{2} \quad (16)$$

In the ordinary formulation of non-relativistic quantum mechanics, the Hamiltonian \mathbf{H} of an ideal free particle (considered as a particle with $\mathcal{B} = 0$ and then $\mathcal{A} \rightarrow +\infty$) of mass m is

$$\mathbf{H} = \frac{(\mathbf{p})^2}{2m} \quad (17)$$

\mathbf{H} and \mathbf{p} has the same stationary time-dependent eigenstates $|\psi(t)\rangle$ which correspond, in coordinate representation, to basic wavefunctions that are plane waves.

$$\langle x | \psi(t) \rangle = \psi(x, t) = C e^{-\frac{i}{\hbar}(\varepsilon t - p x)} \quad (18)$$

the eigenvalues equations are

$$\begin{cases} \mathbf{p}|\psi(t)\rangle = p|\psi(t)\rangle \\ \mathbf{H}|\psi(t)\rangle = \varepsilon|\psi(t)\rangle = \frac{(\mathbf{p})^2}{2m}|\psi(t)\rangle \end{cases} \quad (19)$$

From the relations (17)–(19), we may remark the following inconveniences: according to the probabilistic interpretation of quantum mechanics, the square $|\psi(x, t)|^2$ of the wavefunction in (18) should be interpreted as a probability density. But since this quantity which is equal to $|C|^2$ is a constant, it cannot be normalized as a probability density for $x \in]-\infty, +\infty[$. In other words the wavefunction (18) is, in general, not rigorously compatible with the probabilistic interpretation of quantum mechanic. It follows also that the concept of coordinate and momentum variances, which are at the core of the uncertainty relation (16) are ill defined for this wavefunction (18): we may say that the wavefunction (18) is not ‘rigorously compatible’ with the uncertainty relation (16). Explicitly, it may be considered as corresponding to the ‘singular limit’: $\mathcal{B} \rightarrow 0$ and $\mathcal{A} \rightarrow +\infty$.

A solution to the above problems can be obtained if we consider that the particle is described by a wave packet ($\mathcal{B} > 0$) but not by the plane wave (18). We should find a general expression of the Hamiltonian of the particle, with $\mathcal{B} > 0$, and which leads to the particular case described by the relations (17)–(19) in the ideal limit $\mathcal{B} \rightarrow 0$. We may suppose that the energy of a nonrelativistic quantum particle of mass m with respectively a momentum and coordinate mean values $\langle p \rangle$ and $\langle x \rangle$ is composed of two parts: a classical-like kinetic energy $\frac{\langle p \rangle^2}{2m}$ corresponding to the mean value $\langle p \rangle$ of the momentum and a quantum kinetic energy which corresponds to quantum fluctuation of momentum ($\mathcal{B} > 0$). This hypothesis can be described more explicitly if we consider that the Hamiltonian of a real nonrelativistic quantum free particle is exactly similar to the Hamiltonian of a quantum harmonic oscillator with nonzero momentum and coordinate mean values. The expression of this Hamiltonian operator is given in the appendix A by the relations (A1) or (A15)

$$H = \frac{\langle p \rangle^2}{2m} + \frac{\mathcal{B}^+}{m} = \frac{\langle p \rangle^2}{2m} + \frac{\mathcal{B}}{m}(\mathbf{z}^\dagger \mathbf{z} + \mathbf{z} \mathbf{z}^\dagger) = \frac{\langle p \rangle^2}{2m} + \frac{\mathcal{B}}{m}(2\mathbf{z}^\dagger \mathbf{z} + 1) \tag{20}$$

As it will be shown through the next paragraphs, the Hamiltonian operator in (20) solve the inconveniences related to the Hamiltonian operator in (17) and the corresponding eigenfunction (18) that was identified previously. The Hamiltonian (20) is explicitly related to an invariant quadratic operator associated to LCTs.

In the section 4, a new model of ideal gas is considered using the three-dimensional generalization of the Hamiltonian (20). It leads to the establishment of a relation between the momentum variance \mathcal{B} and thermodynamic variables and a new form of the ideal gas state equation with new quantum corrections. This new quantum form of the ideal gas state equation may be used to perform experimental test of the validity of the hypothesis which corresponds to the expression (20).

As already said, the relation (20) corresponds to a new model of a free quantum particle that is exactly considered as a harmonic oscillator with nonzero mean values of the momentum and coordinates: this model is fully described in the appendix A. The evolution equation of the particle is given by the relation (A4) in this appendix and the corresponding basic stationary self-consistent solutions are given by the relations (A10) or (A14)

$$|\psi_n(t)\rangle = e^{-\frac{i}{\hbar}\varepsilon_n t} |n, \langle p \rangle, \langle x \rangle, \mathcal{B}\rangle = e^{-\frac{i}{\hbar}\varepsilon_n t} |n, \langle z \rangle\rangle \tag{21}$$

with ε_n an eigenvalue of the Hamiltonian operator. Its expression is given by the relation (A11)

$$\varepsilon_n = \frac{(\langle \psi_n(t) | \mathbf{p} | \psi_n(t) \rangle)^2}{2m} + (2n + 1) \frac{\mathcal{B}}{m} = \frac{\langle p \rangle^2}{2m} + (2n + 1) \frac{\mathcal{B}}{m} \tag{22}$$

Both the set $\{|\psi_n(t)\rangle\}$ and $\{|n, \langle z \rangle\rangle\}$ are sets of eigenstates of the Hamiltonian operators associated with the eigenvalues ε_n and are orthonormal basis of the states space of the particle. It can also be checked easily that the set of the coordinate wavefunctions $\{\langle x | n, \langle z \rangle \rangle\}$ is also an orthonormal basis of the L^2 space of square integral functions so it follows that for any state $|\psi\rangle$ of a particle (which is free or not) we can have the decomposition

$$|\psi\rangle = \sum_n |n, \langle z \rangle\rangle \langle n, \langle z \rangle | \psi \rangle = \sum_n \Psi^n(\langle p \rangle, \langle x \rangle, \mathcal{B}) |n, \langle z \rangle\rangle \tag{23}$$

with the wavefunction $\Psi^n(\langle p \rangle, \langle x \rangle, \mathcal{B}) = \langle n, \langle z \rangle | \psi \rangle$. $|\Psi^n(\langle p \rangle, \langle x \rangle, \mathcal{B})|^2 = \langle n, \langle z \rangle | \psi \rangle^2$ is the probability to find the particle in the state $|n, \langle z \rangle\rangle$ when the state is $|\psi\rangle$.

3.2. Phase space representation and quantum phase space

Following the relation (A13) in the appendix A, a state $|\langle z \rangle\rangle = |0, \langle z \rangle\rangle = |0, \langle p \rangle, \langle x \rangle, \mathcal{B}\rangle$ is an eigenstate of the operator $\mathbf{z} = \mathbf{p} + \frac{2i}{\hbar}\mathcal{B}\mathbf{x}$

$$\mathbf{z}|\langle z \rangle\rangle = \langle z \rangle |\langle z \rangle\rangle = \left(\langle p \rangle + \frac{2i}{\hbar}\mathcal{B}\langle x \rangle \right) |\langle z \rangle\rangle \tag{24}$$

A state $|\langle z \rangle\rangle$ can be explicitly and completely characterized by the corresponding wavefunction $\langle x | \langle z \rangle \rangle$ in coordinate representation. It can be deduced as a particular case of the relation (A5) in the appendix A for $n = 0$

$$\langle x | \langle z \rangle \rangle = \langle x | 0, \langle p \rangle, \langle x \rangle, \mathcal{B} \rangle = \left(\frac{2\mathcal{B}}{\pi\hbar^2} \right)^{1/4} e^{-\frac{\mathcal{B}}{\hbar^2}(x-\langle x \rangle)^2 + \frac{i}{\hbar}\langle p \rangle x} = \left(\frac{1}{2\pi\mathcal{A}} \right)^{1/4} e^{-\frac{1}{4\mathcal{A}}(x-\langle x \rangle)^2 + \frac{i}{\hbar}\langle p \rangle x} \tag{25}$$

with $\mathcal{A} = \frac{(\hbar)^2}{4\mathcal{B}}$ the coordinate statistical variance when the particle is in a state $|\langle z \rangle\rangle$.

The set $\{|\langle z \rangle\rangle\}$ of the eigenstates of \mathbf{z} is not an orthonormal basis (\mathbf{z} is not hermitian). Explicitly, it can be checked that we have the scalar product

$$\langle\langle z \rangle | \langle z \rangle'\rangle = \int \langle\langle z \rangle | x \rangle \langle x | \langle z \rangle'\rangle dx = e^{-\frac{(\langle p \rangle - \langle p' \rangle)^2}{8\mathcal{B}} - \frac{(\langle x \rangle - \langle x' \rangle)^2}{8\mathcal{A}} - i\frac{(\langle p \rangle - \langle p' \rangle)(\langle x \rangle + \langle x' \rangle)}{2\hbar}} \tag{26}$$

However, it can be shown that we have the dyadic decomposition ($\hbar = 2\pi\hbar$ is the Planck constant)

$$\iint |\langle z \rangle\rangle \langle\langle z \rangle| \frac{d\langle p \rangle d\langle x \rangle}{h} = I \tag{27}$$

in which I is the identity operator. It follows from the relation (27) that for any possible quantum state $|\psi\rangle$ of a particle and for any operator A , we have the decompositions

$$|\psi\rangle = I|\psi\rangle = \iint \langle\langle z \rangle\rangle \langle\langle z \rangle | \psi \rangle \frac{d\langle p \rangle d\langle x \rangle}{h} = \iint \Psi^0(\langle p \rangle, \langle x \rangle, \mathcal{B}) \langle\langle z \rangle\rangle \frac{d\langle p \rangle d\langle x \rangle}{h} \tag{28}$$

$$A = IA = \iint \iint \langle\langle z \rangle\rangle \langle\langle z \rangle | A | \langle z \rangle'\rangle \langle\langle z \rangle'| \frac{d\langle p \rangle d\langle x \rangle}{h} \frac{d\langle p \rangle' d\langle x \rangle'}{h} \tag{29}$$

The relations (28) and (29) can be considered as defining ‘phase spaces representations’ of a state $|\psi\rangle$ and of a quantum operator A : they define a ‘phase space representation of quantum mechanics’. In this point of view, a ‘quantum phase space’ is, according to relations (27)–(29), to be identified with the set $\{\langle p \rangle, \langle x \rangle\}$ for a given value of \mathcal{B} .

We have for the wavefunction $\Psi^0(\langle p \rangle, \langle x \rangle, \mathcal{B}) = \langle\langle z \rangle | \psi \rangle$ in (28) the normalization relation

$$\iint \langle\langle z \rangle | \psi \rangle^2 \frac{d\langle p \rangle d\langle x \rangle}{h} = 1 \Leftrightarrow \iint \langle\langle z \rangle | \psi \rangle^2 d\langle p \rangle d\langle x \rangle = h \tag{30}$$

It can be remarked that the presence of the Planck constant $\hbar = 2\pi\hbar$ in the relations (27)–(30) may be related to the important fact that in statistical mechanics the surface of a ‘phase space elementary cell’ should be taken to be equal to h . The normalization relation (30) can be considered as saying that $\Psi^0(\langle p \rangle, \langle x \rangle, \mathcal{B}) = |\langle\langle z \rangle | \psi \rangle|^2$ corresponds to a description of the ‘spreadness’ of the particle microstate $|\psi\rangle$ on the quantum phase space.

Remark: we may consider the following operators denoted \mathfrak{P} , \mathfrak{x} and \mathfrak{Q}^+ that have been introduced in the references [10, 11]. They are called respectively reduced momentum operator, reduced coordinate operator and reduced momentum dispersion operator

$$\begin{cases} \mathfrak{P} = \frac{1}{\sqrt{2}}(\mathfrak{z}^\dagger + \mathfrak{z}) = \frac{i}{\sqrt{2\mathcal{B}}}(\mathfrak{p} - \langle p \rangle) = \frac{i}{\hbar}\sqrt{2\mathcal{A}}(\mathfrak{p} - \langle p \rangle) \\ \mathfrak{x} = \frac{i}{\sqrt{2}}(\mathfrak{z}^\dagger - \mathfrak{z}) = \frac{1}{i\sqrt{2\mathcal{A}}}(\mathfrak{x} - \langle x \rangle) = \frac{1}{i\hbar}\sqrt{2\mathcal{B}}(\mathfrak{x} - \langle x \rangle) \\ \mathfrak{Q}^+ = \frac{\mathfrak{Q}^+}{4\mathcal{B}} = \frac{1}{4}(\mathfrak{z}^\dagger\mathfrak{z} + \mathfrak{z}\mathfrak{z}^\dagger) = -\frac{1}{4}[(\mathfrak{P})^2 + (\mathfrak{x})^2] \end{cases} \tag{31}$$

The operators \mathfrak{z}^\dagger and \mathfrak{z} in these relations being the ladder operator defined in the appendix A by the relation (A15).

As shown in the reference [11], the generalization of these operators plays an important role in the study of general multidimensional LCTs. The generalization of the operator \mathfrak{Q}^+ , in particular, is an invariant quadratic operator associated to LCTs.

3.3. Invariant quadratic operator associated to general monodimensional LCTs

The LCTs considered in the relation (2) is a particular case of monodimensional LCTs. general ones satisfy the following relations (algebraic covariant and contravariant index are introduced)

$$\begin{cases} \mathfrak{p}'_1 = \mathfrak{a}_1^1 \mathfrak{p}_1 + \mathfrak{b}_1^1 \mathfrak{x}_1 \\ \mathfrak{x}'_1 = \mathfrak{c}_1^1 \mathfrak{p}_1 + \mathfrak{d}_1^1 \mathfrak{x}_1 \\ [\mathfrak{p}'_1, \mathfrak{x}'_1] = [\mathfrak{p}_1, \mathfrak{x}_1] = -i\hbar \end{cases} \Leftrightarrow \mathfrak{a}_1^1 \mathfrak{d}_1^1 - \mathfrak{b}_1^1 \mathfrak{c}_1^1 = 1 \Leftrightarrow \begin{pmatrix} \mathfrak{a}_1^1 & \mathfrak{c}_1^1 \\ \mathfrak{b}_1^1 & \mathfrak{d}_1^1 \end{pmatrix} \in SL(2) \tag{32}$$

in which $SL(2)$ is the Special Linear group.

The identification of the commutation relation $[\mathfrak{p}'_1, \mathfrak{x}'_1] = [\mathfrak{p}_1, \mathfrak{x}_1] = -i\hbar$ as a monodimensional particular case of the general multidimensional form $[\mathfrak{p}_\mu, \mathfrak{x}_\nu] = i\hbar\eta_{\mu\nu}$ considered in the reference [11] permits to identify ‘the metric’ $\eta = \eta_{11} = -1$.

As shown in [11], the general relation for a $2D \times 2D$ matrix $\begin{pmatrix} \mathfrak{a} & \mathfrak{c} \\ \mathfrak{b} & \mathfrak{d} \end{pmatrix}$ corresponding to an LCT in a D -dimensional pseudo-Euclidian space with a metric η with signature (D_+, D_-) is

$$\begin{pmatrix} \mathbf{a} & \mathbf{c} \\ \mathbf{b} & \mathbf{d} \end{pmatrix}^T \begin{pmatrix} 0 & \eta \\ -\eta & 0 \end{pmatrix} \begin{pmatrix} \mathbf{a} & \mathbf{c} \\ \mathbf{b} & \mathbf{d} \end{pmatrix} = \begin{pmatrix} 0 & \eta \\ -\eta & 0 \end{pmatrix} \tag{33}$$

and $\begin{pmatrix} \mathbf{a} & \mathbf{c} \\ \mathbf{b} & \mathbf{d} \end{pmatrix}$ belongs then to a symplectic group $Sp(2D_+, 2D_-)$. For the space associated to the monodimensional LCT in (32), the signature is (0, 1) i.e $\eta = \eta_{11} = -1$ and we have the explicit equivalence

$$\begin{pmatrix} \mathbf{a}_1^1 & \mathbf{c}_1^1 \\ \mathbf{b}_1^1 & \mathbf{d}_1^1 \end{pmatrix}^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1^1 & \mathbf{c}_1^1 \\ \mathbf{b}_1^1 & \mathbf{d}_1^1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \iff \mathbf{a}_1^1 \mathbf{d}_1^1 - \mathbf{b}_1^1 \mathbf{c}_1^1 = 1 \tag{34}$$

The relation (34) corresponds to the group isomorphism $Sp(0,2) \cong SL(2)$.

The LCT (32) can be considered as a transformation which affects the momentum and coordinate operators (and any operator depending on them) but does not affect the quantum state of a particle : this is the point of view considered in the [11, 12] in which a general multidimensional LCT is considered as describing a change of observational reference frame.

From now, for sake of simplicity, we will use the natural system unit (in which we have respectively in particular for the reduced Planck constant and the speed of light : $\hbar = 1$ and $c = 1$). Let $|\psi\rangle$ be the state of a particle. We may introduce the following mean values and statistical variance-covariance

$$\begin{cases} \langle x_1 \rangle = \langle \psi | \mathbf{x}_1 | \psi \rangle = -\langle \psi | \mathbf{x}^1 | \psi \rangle = -\langle x^1 \rangle \\ \langle p_1 \rangle = \langle \psi | \mathbf{p}_1 | \psi \rangle = -\langle \psi | \mathbf{p}^1 | \psi \rangle = -\langle p^1 \rangle \\ \mathcal{X}_{11} = \langle \psi | (\mathbf{x}_1 - \langle x_1 \rangle)^2 | \psi \rangle = -\mathcal{X}_1^1 = \mathcal{X}^{11} \\ \mathcal{P}_{11} = \langle \psi | (\mathbf{p}_1 - \langle p_1 \rangle)^2 | \psi \rangle = -\mathcal{P}_1^1 = \mathcal{P}^{11} \\ \varrho_{11}^{\times} = \langle \psi | (\mathbf{p}_1 - \langle p_1 \rangle)(\mathbf{x}_1 - \langle x_1 \rangle) | \psi \rangle \\ \varrho_{11}^{\times} = \langle \psi | (\mathbf{x}_1 - \langle x_1 \rangle)(\mathbf{p}_1 - \langle p_1 \rangle) | \psi \rangle \\ \varrho_{11} = \frac{1}{2}(\varrho_{11}^{\times} + \varrho_{11}^{\times}) \end{cases} \tag{35}$$

ϱ_{11}^{\times} , ϱ_{11}^{\times} and ϱ_{11} can be both considered as some kind of momentum-coordinate statistical covariances (codispersions). But they are different because the operators \mathbf{p} and \mathbf{x} do not commute. However, it can be deduced from the commutation relation $[\mathbf{p}_1, \mathbf{x}_1] = -i$ that we have the relations

$$\varrho_{11}^{\times} - \varrho_{11}^{\times} = -i \Rightarrow \begin{cases} \varrho_{11}^{\times} = \varrho_{11}^{\times} - i = \varrho_{11} - \frac{i}{2} \\ \varrho_{11}^{\times} = \varrho_{11}^{\times} + i = \varrho_{11} + \frac{i}{2} \\ \varrho_{11} = \frac{1}{2}(\varrho_{11}^{\times} + \varrho_{11}^{\times}) = \varrho_{11}^{\times} + \frac{i}{2} = \varrho_{11}^{\times} - \frac{i}{2} \end{cases} \tag{36}$$

Under the action of the LCT (32), the mean values and statistical variance-covariance become

$$\begin{cases} \langle x'_1 \rangle = \langle \psi | \mathbf{x}'_1 | \psi \rangle = -\langle \psi | \mathbf{x}'^1 | \psi \rangle = -\langle x'^1 \rangle \\ \langle p'_1 \rangle = \langle \psi | \mathbf{p}'_1 | \psi \rangle = -\langle \psi | \mathbf{p}'^1 | \psi \rangle = -\langle p'^1 \rangle \\ \mathcal{X}'_{11} = \langle \psi | (\mathbf{x}'_1 - \langle x'_1 \rangle)^2 | \psi \rangle = -\mathcal{X}'^1_1 = \mathcal{X}'^{11} \\ \mathcal{P}'_{11} = \langle \psi | (\mathbf{p}'_1 - \langle p'_1 \rangle)^2 | \psi \rangle = -\mathcal{P}'^1_1 = \mathcal{P}'^{11} \\ \varrho'^{\times}_{11} = \langle \psi | (\mathbf{p}'_1 - \langle p'_1 \rangle)(\mathbf{x}'_1 - \langle x'_1 \rangle) | \psi \rangle \\ \varrho'^{\times}_{11} = \langle \psi | (\mathbf{x}'_1 - \langle x'_1 \rangle)(\mathbf{p}'_1 - \langle p'_1 \rangle) | \psi \rangle \\ \varrho'_{11} = \frac{1}{2}(\varrho'^{\times}_{11} + \varrho'^{\times}_{11}) \end{cases} \tag{37}$$

Using the relations (32), (35) and (37), we can deduce the laws of transformations

$$\langle p'_1 \rangle \langle x'_1 \rangle = \langle p_1 \rangle \langle x_1 \rangle \begin{pmatrix} \mathbf{a}_1^1 & \mathbf{c}_1^1 \\ \mathbf{b}_1^1 & \mathbf{d}_1^1 \end{pmatrix} \tag{38}$$

$$\begin{pmatrix} \mathcal{P}'_{11} & \varrho'_{11} \\ \varrho'_{11} & \mathcal{X}'_{11} \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1^1 & \mathbf{c}_1^1 \\ \mathbf{b}_1^1 & \mathbf{d}_1^1 \end{pmatrix}^T \begin{pmatrix} \mathcal{P}_{11} & \varrho_{11} \\ \varrho_{11} & \mathcal{X}_{11} \end{pmatrix} \begin{pmatrix} \mathbf{a}_1^1 & \mathbf{c}_1^1 \\ \mathbf{b}_1^1 & \mathbf{d}_1^1 \end{pmatrix} \tag{39}$$

Then, the following invariants can be deduced using the relation (34)

- An invariant scalar which is the determinant of the matrix $\begin{pmatrix} \mathcal{P}_{11} & \varrho_{11} \\ \varrho'_{11} & \mathcal{X}_{11} \end{pmatrix}$ that we may call, following the reference [11], momentum–coordinate variance-covariance matrix

$$\begin{vmatrix} \mathcal{P}'_{11} & \varrho'_{11} \\ \varrho'_{11} & \mathcal{X}'_{11} \end{vmatrix} = \mathcal{P}'_{11}\mathcal{X}'_{11} - (\varrho'_{11})^2 = \mathcal{P}_{11}\mathcal{X}_{11} - (\varrho_{11})^2 = \begin{vmatrix} \mathcal{P}_{11} & \varrho_{11} \\ \varrho_{11} & \mathcal{X}_{11} \end{vmatrix} \tag{40}$$

- An invariant quadratic operator that we may denote Ξ^+

$$\Xi^+ = \frac{1}{2}((p_1 - \langle p_1 \rangle) \quad (x_1 - \langle x_1 \rangle)) \begin{pmatrix} \mathcal{X}_{11} & -\varrho_{11} \\ -\varrho_{11} & \mathcal{P}_{11} \end{pmatrix} ((p_1 - \langle p_1 \rangle) \quad (x_1 - \langle x_1 \rangle))^T \tag{41}$$

The notation Ξ^+ is chosen because this invariant quadratic operator in the relation (41) can be considered as a generalization of the reduced momentum dispersion operator in (31). In fact, in the limit $\varrho_{11} = 0$, it can be checked that the two operators is exactly equal.

Using the coordinate representation, it can be shown that the eigenstate of the invariant quadratic operator Ξ^+ in (41) corresponding to its lowest eigenvalue is a state, that we may denote $|\langle z_1 \rangle\rangle$, corresponding to the coordinate wavefunction of the form

$$\langle x^1 | \langle z_1 \rangle \rangle = e^{iK} \left(\frac{1}{2\pi\mathcal{X}_{11}} \right)^{1/4} e^{-\mathcal{B}_{11}(x^1 - \langle x^1 \rangle)^2 - i\langle p_1 \rangle x^1} \tag{42}$$

in which e^{iK} is an unitary complex number (K is real number). $\langle p_1 \rangle$, $\langle x^1 \rangle$ and \mathcal{X}_{11} are the mean values and statistical variances, as defined in (35), associated to the state $|\psi\rangle = |\langle z_1 \rangle\rangle$ itself. \mathcal{B}_{11} is a complex parameter given by the relation

$$\mathcal{B}_{11} = -i \frac{\varrho_{11}^\times}{2\mathcal{X}_{11}} = \frac{1}{4\mathcal{X}_{11}} - i \frac{\varrho_{11}}{2\mathcal{X}_{11}} \tag{43}$$

with ϱ_{11}^\times and ϱ_{11} the momentum-coordinate covariances, as defined in (35), associated to the state $|\psi\rangle = |\langle z_1 \rangle\rangle$. If we denoted also \mathcal{P}_{11} the momentum statistical variance for this state $|\psi\rangle = |\langle z_1 \rangle\rangle$, it can be checked, using (42), that for this particular case the LCT-invariant scalar (40) is exactly given by relation

$$\mathcal{P}'_{11}\mathcal{X}'_{11} - (\varrho'_{11})^2 = \mathcal{P}_{11}\mathcal{X}_{11} - (\varrho_{11})^2 = \frac{1}{4} \tag{44}$$

And this invariant scalar is exactly the lowest eigenvalue of the LCT-invariant quadratic operator Ξ^+ in (41) i.e. corresponding to the eigenstate $|\langle z_1 \rangle\rangle$

$$\Xi^+ |\langle z_1 \rangle \rangle = [\mathcal{P}_{11}\mathcal{X}_{11} - (\varrho_{11})^2] |\langle z_1 \rangle \rangle = \frac{1}{4} |\langle z_1 \rangle \rangle \tag{45}$$

The notation $|\langle z_1 \rangle\rangle$ is chosen for the state corresponding to the wavefunction in (42) because it can be considered as a generalization of the state $|\langle z \rangle\rangle$ corresponding to the wavefunction in (25). The relation (42) is a generalization of (25): (33) is covariant under the action of the particular LCT (2) but (42) is covariant under the action of the general monodimensional LCT (32). The parameters \mathcal{B} in (25) is a real number and is equal to the momentum statistical variances. But the parameter \mathcal{B}_{11} in (42) is a complex number. \mathcal{B}_{11} is related to the coordinate statistical variance and momentum-coordinate covariance by the relation (43). And it can be deduced from (43) and (45) that the relation between \mathcal{B}_{11} and the momentum statistical variance \mathcal{P}_{11} is

$$\mathcal{P}_{11} = \mathcal{B}_{11} \left(1 + i \frac{\varrho_{11}}{2} \right) \tag{46}$$

In the limit $\varrho_{11} = 0$, we have as expected $\mathcal{P}_{11} = \mathcal{B}_{11}$.

Given the fact that the operator Ξ^+ in the relation (41) and its lowest eigenvalue are LCT-invariants, it follows that the LCT-transforms of the wavefunction (42) should be of the form

$$\langle x^{1'} | \langle z_1 \rangle \rangle = e^{iK'} \left(\frac{1}{2\pi\mathcal{X}'_{11}} \right)^{1/4} e^{-\mathcal{B}'_{11}(x^{1'} - x^v)^2 - ip'_1 x^v} \tag{47}$$

in which, $e^{iK'}$ is a unitary complex number (K' is real).

3.4. Ladder operators and momentum dispersion operator

Using coordinate representation and the wavefunctions in (42) and (47), it can be shown that the state $|\langle z_1 \rangle\rangle$ is a common eigenstate of the operator $z_1 = p_1 + 2i\mathcal{B}_{11}x^1$ and its LCT-transform $z_1' = p_1' + 2i\mathcal{B}'_{11}x'^1$. The corresponding eigenvalue equations are

$$\begin{cases} z_1|\langle z_1 \rangle\rangle = \langle z_1 \rangle|\langle z_1 \rangle\rangle = \left(\langle p_1 \rangle + \frac{2i}{\hbar}\mathcal{B}_{11}\langle x^1 \rangle \right)|\langle z_1 \rangle\rangle \\ z_1'|\langle z_1 \rangle\rangle = \langle z_1' \rangle|\langle z_1 \rangle\rangle = \left(\langle p_1' \rangle + \frac{2i}{\hbar}\mathcal{B}'_{11}\langle x'^1 \rangle \right)|\langle z_1' \rangle\rangle \end{cases} \quad (48)$$

we have the 'covariant commutation relations'

$$[z_1, z_1^\dagger] = \frac{1}{\mathcal{X}_{11}} \quad [z_1', z_1'^\dagger] = \frac{1}{\mathcal{X}'_{11}} \quad (49)$$

Then, as generalization of the relations (A15) in the appendix A and (31), we introduce the following ladder operators $\mathfrak{z}_1, \mathfrak{z}_1^\dagger$ and reduced operators $\mathfrak{p}_1, \mathfrak{x}_1$ that are covariants under the action of the general monodimensional LCTs (32)

$$\begin{cases} \mathfrak{z}_1 = a_1^1(z_1 - \langle z_1 \rangle) = \frac{1}{\sqrt{2}}(\mathfrak{p}_1 + i\mathfrak{x}_1) \\ \mathfrak{z}_1^\dagger = a_1^{1*}(z_1^\dagger - \langle z_1^\dagger \rangle) = \frac{1}{\sqrt{2}}(\mathfrak{p}_1^\star - i\mathfrak{x}_1^\star) \end{cases} \quad [\mathfrak{z}_1, \mathfrak{z}_1^\dagger] = 1 \quad (50)$$

with

$$\begin{cases} \mathfrak{p}_1 = \sqrt{2} a_1^1(p_1 - \langle p_1 \rangle) - \sqrt{2} c_1^1(x_1 - \langle x_1 \rangle) & \mathfrak{x}_1 = \sqrt{2} \ell_1^1(x_1 - \langle x_1 \rangle) \\ \mathfrak{p}_1^\star = \sqrt{2} a_1^{1*}(p_1 - \langle p_1 \rangle) - \sqrt{2} c_1^{1*}(x_1 - \langle x_1 \rangle) & \mathfrak{x}_1^\star = \sqrt{2} \ell_1^{1*}(x_1 - \langle x_1 \rangle) \end{cases} \quad (51)$$

in which the parameters a_1^1, ℓ_1^1 and c_1^1 introduced in (50) and (51) are related to the momentum-coordinate variance-covariance by the relation

$$\begin{cases} (a_1^1)^2 = \mathcal{X}_1^1 = -\mathcal{X}_{11} \Rightarrow a_1^1 = i\sqrt{\mathcal{X}_{11}} \\ \ell_1^1 = \frac{1}{2a_1^1} = \frac{-i}{2\sqrt{\mathcal{X}_{11}}} \\ c_1^1 = \frac{\varrho_{11}}{a_1^1} = \frac{-i\varrho_{11}}{\sqrt{\mathcal{X}_{11}}} \end{cases} \quad (52)$$

It may be noticed that unlike the operators p_1 and x_1 , the reduced operators \mathfrak{p}_1 and \mathfrak{x}_1 in (51) are not, in general, hermitians: in fact, the parameters a_1^1, ℓ_1^1 and c_1^1 can have complex values, like in the relation (52), when the metric is not positive definite.

Using the relation (44), the relation (52) can also be put in the matrix form

$$\begin{pmatrix} \mathcal{P}_{11} & \varrho_{11} \\ \varrho_{11} & \mathcal{X}_{11} \end{pmatrix} = \begin{pmatrix} \ell_1^1 & 0 \\ c_1^1 & a_1^1 \end{pmatrix}^T \begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix} \begin{pmatrix} \ell_1^1 & 0 \\ c_1^1 & a_1^1 \end{pmatrix} = \begin{pmatrix} \ell_1^1 & 0 \\ c_1^1 & a_1^1 \end{pmatrix}^T \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \ell_1^1 & 0 \\ c_1^1 & a_1^1 \end{pmatrix} \quad (53)$$

and we have the properties

$$\begin{pmatrix} \ell_1^1 & 0 \\ c_1^1 & a_1^1 \end{pmatrix}^{-1} = 2 \begin{pmatrix} a_1^1 & 0 \\ -c_1^1 & \ell_1^1 \end{pmatrix} \quad (54)$$

Using the relations (32), (38), (51), (53) and (54), the law of transformation of the reduced operators \mathfrak{p}_1 and \mathfrak{x}_1 defined in (53) can be written in the matricial form

$$(\mathfrak{p}_1' \ \mathfrak{x}_1') = (\mathfrak{p}_1 \ \mathfrak{x}_1) \begin{pmatrix} \Pi_1^1 & \Xi_1^1 \\ \Theta_1^1 & \Lambda_1^1 \end{pmatrix} \quad (55)$$

with

$$\begin{pmatrix} \Pi_1^1 & \Xi_1^1 \\ \Theta_1^1 & \Lambda_1^1 \end{pmatrix} = \begin{pmatrix} \Pi_1^1 & -\Theta_1^1 \\ \Theta_1^1 & \Pi_1^1 \end{pmatrix} = 2 \begin{pmatrix} \ell_1^1 & 0 \\ c_1^1 & a_1^1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1^1 & \mathbf{c}_1^1 \\ \mathbf{b}_1^1 & \mathbf{d}_1^1 \end{pmatrix} \begin{pmatrix} a_1^1 & 0 \\ -c_1^1 & \ell_1^1 \end{pmatrix} \quad (56)$$

Using the relation (34), it can be deduced that the matrix $\begin{pmatrix} \Pi_1^1 & \Xi_1^1 \\ \Theta_1^1 & \Lambda_1^1 \end{pmatrix}$ satisfies the relation

$$\begin{cases} \begin{pmatrix} \Pi_1^1 & \Xi_1^1 \\ \Theta_1^1 & \Lambda_1^1 \end{pmatrix}^T \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \Pi_1^1 & \Xi_1^1 \\ \Theta_1^1 & \Lambda_1^1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\ \begin{pmatrix} \Pi_1^1 & \Xi_1^1 \\ \Theta_1^1 & \Lambda_1^1 \end{pmatrix}^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Pi_1^1 & \Xi_1^1 \\ \Theta_1^1 & \Lambda_1^1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{cases} \tag{57}$$

The relation (57) means that $\begin{pmatrix} \Pi_1^1 & \Xi_1^1 \\ \Theta_1^1 & \Lambda_1^1 \end{pmatrix} = \begin{pmatrix} \Pi_1^1 & -\Theta_1^1 \\ \Theta_1^1 & \Pi_1^1 \end{pmatrix}$ belongs to the group intersection

$$Sp(0, 2) \cap O(0, 2) \cong SO(0, 2) \cong SO(2) \tag{58}$$

From the relations (50) we have for the law of transformation of \mathfrak{z}_1

$$\mathfrak{z}'_1 = \mathfrak{z}_1 \Omega_1^1 = \mathfrak{z}_1 (\Pi_1^1 - i\Theta_1^1) \tag{59}$$

with $\Omega_1^1 = \Pi_1^1 - i\Theta_1^1$ satisfying the relation $\Omega_1^{1*}(-1)^1 = -1$ i.e Ω_1^1 is an element of the group $U(0, 1) \cong U(1)$.

Using the relations (55), (57) and (59) it can be checked that the following quadratic operator is invariant under the action of the LCT (32) (with $\eta^{11} = \eta_{11} = -1$)

$$\Xi^+ = \frac{1}{4} \eta^{11} [(\mathfrak{p}_1)^2 + (\mathfrak{x}_1)^2] = \frac{1}{4} (\mathfrak{z}_1^\dagger \mathfrak{z}_1 + \mathfrak{z}_1 \mathfrak{z}_1^\dagger) = \frac{1}{4} (2\mathfrak{z}_1^\dagger \mathfrak{z}_1 + 1) \tag{60}$$

and using the relations (50), (51) and (53) it can be remarked that this operator is exactly the quadratic invariant operator in (41). An expression of a general eigenstate, denoted $|n_1, \langle z_1 \rangle\rangle$, of the LCT invariant operator can be deduced from the relation (48), (50) and (60)

$$|n_1, \langle z_1 \rangle\rangle = \frac{(\mathfrak{z}_1^\dagger)^{n_1}}{\sqrt{n_1!}} | \langle z_1 \rangle \rangle \tag{61}$$

The corresponding eigenvalue equation is

$$\Xi^+ |n_1, \langle z_1 \rangle\rangle = \frac{1}{4} (2n_1 + 1) |n_1, \langle z_1 \rangle\rangle \tag{62}$$

and we have

$$\begin{cases} [\Xi^+, \mathfrak{z}_1] = -\mathfrak{z}_1 \Rightarrow \mathfrak{z}_1 |n_1, \langle z_1 \rangle\rangle = \sqrt{n_1} |n_1 - 1, \langle z_1 \rangle\rangle \\ [\Xi^+, \mathfrak{z}_1^\dagger] = \mathfrak{z}_1^\dagger \Rightarrow \mathfrak{z}_1^\dagger |n_1, \langle z_1 \rangle\rangle = \sqrt{n_1 + 1} |n_1 + 1, \langle z_1 \rangle\rangle \end{cases} \tag{63}$$

The relations (59)–(63) show that \mathfrak{z}_1 and \mathfrak{z}_1^\dagger are the ladder operators associated to the LCT-invariant quadratic operator Ξ^+ .

From the relations (50)–(54) and (63), we obtain the relation

$$\langle n_1, \langle z_1 \rangle | (\mathfrak{p}_1 - \langle p_1 \rangle)^2 | n_1, \langle z_1 \rangle \rangle = (2n_1 + 1) \mathcal{P}_{11} \tag{64}$$

The momentum statistical variance of the state $|n_1, \langle z_1 \rangle\rangle$ is equal to $(2n_1 + 1) \mathcal{P}_{11}$ with \mathcal{P}_{11} the statistical variance of the state $|0, \langle z_1 \rangle\rangle = | \langle z_1 \rangle \rangle$. Given the relations (62) and (64), we may define the general monodimensional momentum dispersion operator $\mathfrak{D}_{11}^+ = 4\Xi^+ \mathcal{P}_{11}$. The eigenstates of \mathfrak{D}_{11}^+ are also the states $|n_1, \langle z_1 \rangle\rangle$ and the corresponding eigenvalue equation is

$$\mathfrak{D}_{11}^+ |n_1, \langle z_1 \rangle\rangle = 4\Xi^+ \mathcal{P}_{11} |n_1, \langle z_1 \rangle\rangle = (2n_1 + 1) \mathcal{P}_{11} |n_1, \langle z_1 \rangle\rangle \tag{65}$$

The quadratic operator Ξ^+ is LCT-invariant but the momentum dispersion operator \mathfrak{D}_{11}^+ is not invariant because \mathcal{P}_{11} is not invariant (its law of transformation is given by the relation (39)). However, \mathfrak{D}_{11}^+ is LCT-covariant. Under the action of the LCT (32) we have

$$\mathfrak{D}'_{11} = 4\Xi^+ \mathcal{P}'_{11} = \frac{\mathcal{P}'_{11}}{\mathcal{P}_{11}} \mathfrak{D}_{11}^+ \tag{66}$$

A generalization of the Hamiltonian operator in (20) using the LCT-covariant momentum dispersion operator \mathfrak{D}_{11}^+ can be deduced from the relation $\mathfrak{D}_{11}^+ = 4\Xi^+ \mathcal{P}_{11}$

$$H = \frac{\langle p_1 \rangle^2}{2m} + \frac{\mathfrak{Q}_{11}^+}{m} = \frac{\langle p \rangle^2}{2m} + \frac{4\mathfrak{Q}^+ \mathcal{P}_{11}}{m} = \frac{\langle p_1 \rangle^2}{2m} + \frac{\mathcal{P}_{11}}{m} (2\mathfrak{z}_1^\dagger \mathfrak{z}_1 + 1) \quad (67)$$

The eigenvalue equation of this Hamiltonian is

$$H|n_1, \langle z_1 \rangle\rangle = \left[\frac{\langle p_1 \rangle^2}{2m} + (2n_1 + 1) \frac{\mathcal{P}_{11}}{m} \right] |n_1, \langle z_1 \rangle\rangle \quad (68)$$

The eigenstates of this Hamiltonian are also the states $|n_1, \langle z_1 \rangle\rangle$.

Remarks:

- The set $\{|n_1, \langle z_1 \rangle\rangle\}$ (for a fixed $\langle z_1 \rangle$) and $\{|\langle z_1 \rangle\rangle\}$ are respectively an orthonormal basis and an overcomplete frame corresponding to the space state of the particle. They can be used to define a LCT-covariant phase space representation.
- Instead of the operator \mathfrak{Q}^+ in the relation (60), we may consider the simpler quadratic operator

$$\mathfrak{N} = \mathfrak{N}_{11} = \mathfrak{z}_1^\dagger \mathfrak{z}_1 = 2\mathfrak{Q}^+ - \frac{1}{2} \quad (69)$$

which is also an LCT-invariant quadratic operator. $\mathfrak{N} = \mathfrak{N}_{11}$ has the property of a number operator of quasiparticles that we may call ‘dispersion’ (because they are related to the momentum dispersion operator). The ladder operators \mathfrak{z}_1^\dagger and \mathfrak{z}_1 are respectively the creation and annihilation operators associated to these quasiparticles. As these quasiparticles are bosons, we may call \mathfrak{N} a bosonic LCT-invariant quadratic operator (in the section 5, a fermionic LCT-invariant quadratic operator will be also identified).

4. Implication of the quantum momentum variance in thermodynamics

4.1. Relation between quantum momentum variance and thermodynamic variables

Given the relation (20), it may be asked how to relate a quantum statistical parameter like \mathcal{B} to quantity that can be measured. In this section, one of our goals is to show that this parameter can be practically related to thermodynamic variables like pressure P , volume V and temperature T . The formalism of quantum statistical mechanics considered in the appendix B is used for this purpose and the explicit example of a simple ideal Boltzmann gas will be considered as illustration. Our study leads also to the introduction of a quantum correction in the thermodynamic state equation of Boltzmann ideal gas.

According to the relations (B11) or (B13) in the appendix B, the expression of the canonical partition function of a particle described by a Hamiltonian operator which is a three dimensional generalization of (31) is

$$\mathcal{Z} = \frac{1}{8sh^3 \left(\beta \frac{\mathcal{B}}{m} \right)} \quad (70)$$

The relation between the momentum statistical variance \mathcal{B} and thermodynamic variables can be obtained from the semi-classical limit $\mathcal{B} \rightarrow 0$:

- On one hand, at first order of approximation, the expression of \mathcal{Z} in (70) gives

$$\mathcal{Z} = \frac{1}{8sh^3 \left(\beta \frac{\mathcal{B}}{m} \right)} \simeq \left(\frac{m}{2\beta\mathcal{B}} \right)^3 \quad (71)$$

- On the other hand, the semi-classical approximation ($\mathcal{B} \rightarrow 0$) of the integral corresponding to the relation (B13) in the appendix B gives

$$\mathcal{Z} \simeq \frac{1}{h^3} \int e^{-\beta \frac{\vec{p}^2}{2m}} d^3\vec{p} \int_{(V)} d^3\vec{x} = \frac{V}{h^3} \left(\sqrt{\frac{2\pi m}{\beta}} \right)^3 = \frac{V}{(\lambda_{th})^3} \quad (72)$$

with λ_{th} the thermal de Broglie wavelength

$$\lambda_{th} = h \sqrt{\frac{\beta}{2\pi m}} = \frac{h}{\sqrt{2\pi mkT}} \quad (73)$$

From the relations (71) and (72) we can deduce the relation

$$\mathcal{B} = \frac{m\lambda_{th}}{2\beta V^{1/3}} = \frac{\hbar^2}{2V^{1/3}\lambda_{th}} = \frac{\hbar\sqrt{2\pi mkT}}{4\pi V^{1/3}} \quad (74)$$

(74) is the relation which establishes a link between the quantum momentum statistical variance \mathcal{B} and thermodynamics. Using (74), we obtain for the expression of the particle partition function (71) as explicit function of the thermodynamic variables

$$\mathcal{Z} = \frac{1}{8sh^3\left(\beta\frac{\mathcal{B}}{m}\right)} = \frac{1}{8sh^3\left(\frac{\lambda_{th}}{2V^{1/3}}\right)} = \frac{1}{8sh^3\left(\frac{h}{2\sqrt{2\pi mkT}V^{1/3}}\right)} \quad (75)$$

Remarks:

- A comparison between the relations (7) and (74) shows that the particle is equivalent to a harmonic oscillator with effective angular frequency

$$\omega = \frac{2\mathcal{B}}{m\hbar} = \frac{\hbar}{mV^{1/3}\lambda_{th}} = \frac{1}{V^{1/3}}\sqrt{\frac{kT}{2\pi m}} \quad (76)$$

- We have called the approximations corresponding to the relations (71) and (72) ‘semi-classical’ but not classical because we didn’t take there $\mathcal{B} = 0$ but only the first order approximation corresponding to the limit $\mathcal{B} \rightarrow 0$. The approximated expression (72) of the partition function itself maybe called ‘semi-classical’ because it depends on the thermal de Broglie wavelength which is related to \mathcal{B} .

4.2. Quantum correction to the Boltzmann ideal gas thermodynamic equation of state

The quantum corrections that are often considered in the study of ideal gas are the corrections related to their bosonic or fermionic nature: this kind of corrections leads respectively to the Bose-Einstein or Fermi-Dirac statistics. However, the quantum correction that we consider here is not of this kind: it is a correction related to the nonzero value of the momentum statistical variance \mathcal{B} . Explicitly, this correction corresponds to the fact that the partition function that should be used for a particle is (75) instead of (72). The correction related to the bosonic or fermionic nature of particles is not considered here, it may be considered in future study as extension of the present work.

We consider the ideal gas as a set of N indiscernible particles which correspond to a canonical ensemble. The expression of the corresponding partition function denoted \mathcal{Z}_N is

$$\mathcal{Z}_N = \frac{(\mathcal{Z}_1)^N}{N!} = \frac{(\mathcal{Z})^N}{N!} = \frac{1}{8^N N! sh^{3N}\left(\beta\frac{\mathcal{B}}{m}\right)} = \frac{1}{8^N N! sh^{3N}\left(\frac{\lambda_{th}}{2V^{1/3}}\right)} \quad (77)$$

with $\mathcal{Z}_1 = \mathcal{Z}$ the one particle partition function given by the relation (75). The thermodynamical free energy F of the gas is given by the relation

$$F = -kT \ln(\mathcal{Z}_N) \quad (78)$$

and the pressure P is

$$P = -\left(\frac{\partial F}{\partial V}\right)_{T,N} = \left(\frac{\partial [kT \ln(\mathcal{Z}_N)]}{\partial V}\right)_{T,N} \quad (79)$$

The use of the relations (77) and (79) permit to deduce that we have the equation of state

$$PV = NkT \left[\beta\frac{\mathcal{B}}{m} \coth\left(\beta\frac{\mathcal{B}}{m}\right) \right] = NkT \left[\frac{\lambda_{th}}{2V^{1/3}} \coth\left(\frac{\lambda_{th}}{2V^{1/3}}\right) \right] \quad (80)$$

in which *coth* refers to the hyperbolic cotangent function.

In the semi-classical limits $V \rightarrow +\infty$ or $T \rightarrow +\infty$ ($\lambda_{th} \rightarrow 0$) we have

$$\beta\frac{\mathcal{B}}{m} = \frac{\lambda_{th}}{2V^{1/3}} \rightarrow 0 \implies \frac{\lambda_{th}}{2V^{1/3}} \coth\left(\frac{\lambda_{th}}{2V^{1/3}}\right) \rightarrow 1 \quad (81)$$

and the ordinary ideal gas equation of state: $PV = NkT$ is obtained (as an asymptotic limit of the corrected equation of state (80)).

The relation (80) highlight new quantum corrections to the ideal gas thermodynamic equation of state: as already said in the section 3, this new corrections can be considered to conceive experimental tests for the validity of the hypothesis which leads to the expression (20) that is generalized by the relation (B1) in the appendix B.

5. Invariant quadratic operators associated to multidimensional general LCTs

5.1. Bosonic invariant quadratic operators, reduced operators and ladder operators

Let us consider a D -dimensional pseudo-Euclidian space having a metric η with signature (D_+, D_-) . The momenta and coordinates operators are characterized by the Canonical Commutation Relations (CCRs) [11]

$$\begin{cases} [\mathbf{p}_\mu, \mathbf{x}_\nu] = \mathbf{p}_\mu \mathbf{x}_\nu - \mathbf{x}_\nu \mathbf{p}_\mu = i\eta_{\mu\nu} \\ [\mathbf{p}_\mu, \mathbf{p}_\nu] = \mathbf{p}_\mu \mathbf{p}_\nu - \mathbf{p}_\nu \mathbf{p}_\mu = 0 \\ [\mathbf{x}_\mu, \mathbf{x}_\nu] = \mathbf{x}_\mu \mathbf{x}_\nu - \mathbf{x}_\nu \mathbf{x}_\mu = 0 \end{cases} \quad (82)$$

and the general definition of the LCTs is

$$\begin{cases} \mathbf{p}'_\mu = \mathbf{a}'_\mu \mathbf{p}_\nu + \mathbf{b}'_\mu \mathbf{x}_\nu \\ \mathbf{x}'_\mu = \mathbf{c}'_\mu \mathbf{p}_\nu + \mathbf{d}'_\mu \mathbf{x}_\nu \\ [\mathbf{p}'_\mu, \mathbf{x}'_\nu] = [\mathbf{p}_\mu, \mathbf{x}_\nu] = i\eta_{\mu\nu} \\ [\mathbf{p}'_\mu, \mathbf{p}'_\nu] = [\mathbf{p}_\mu, \mathbf{p}_\nu] = 0 \\ [\mathbf{x}'_\mu, \mathbf{x}'_\nu] = [\mathbf{x}_\mu, \mathbf{x}_\nu] = 0 \end{cases} \quad (83)$$

If the $D \times D$ matrices \mathbf{a} , \mathbf{b} , \mathbf{c} , \mathbf{d} and η corresponding to the coefficients \mathbf{a}'_μ , \mathbf{b}'_μ , \mathbf{c}'_μ , \mathbf{d}'_μ and $\eta_{\mu\nu}$ are introduced, the relations in (83) are equivalent to the following matrix relations

$$\begin{cases} \mathbf{a}^T \eta \mathbf{d} - \mathbf{b}^T \eta \mathbf{c} = \eta \\ \mathbf{a}^T \eta \mathbf{b} - \mathbf{b}^T \eta \mathbf{a} = 0 \\ \mathbf{c}^T \eta \mathbf{d} - \mathbf{d}^T \eta \mathbf{c} = 0 \end{cases} \Leftrightarrow \begin{pmatrix} \mathbf{a} & \mathbf{c} \\ \mathbf{b} & \mathbf{d} \end{pmatrix}^T \begin{pmatrix} 0 & \eta \\ -\eta & 0 \end{pmatrix} \begin{pmatrix} \mathbf{a} & \mathbf{c} \\ \mathbf{b} & \mathbf{d} \end{pmatrix} = \begin{pmatrix} 0 & \eta \\ -\eta & 0 \end{pmatrix} \quad (84)$$

According to the relation (84), the $2D \times 2D$ matrix $\begin{pmatrix} \mathbf{a} & \mathbf{c} \\ \mathbf{b} & \mathbf{d} \end{pmatrix}$ belongs to the symplectic group $Sp(2D_+, 2D_-)$ [11, 12]. The LCT group, that we will denote \mathbb{T} like in the [11, 12] can be then identified with $Sp(2D_+, 2D_-)$.

Let $|\psi\rangle$ be a quantum state. As generalization of the relation (35), we may introduce the following momenta and coordinates statistical mean values and variance-covariances

$$\begin{cases} \langle x_\mu \rangle = \langle \psi | \mathbf{x}_\mu | \psi \rangle = \eta_{\mu\nu} \langle \psi | \mathbf{x}'_\nu | \psi \rangle = \eta_{\mu\nu} \langle x'_\nu \rangle \\ \langle p_\mu \rangle = \langle \psi | \mathbf{p}_\mu | \psi \rangle = \eta_{\mu\nu} \langle \psi | \mathbf{p}'_\nu | \psi \rangle = \eta_{\mu\nu} \langle p'_\nu \rangle \\ \mathcal{X}_{\mu\nu} = \langle \psi | (\mathbf{x}_\mu - \langle x_\mu \rangle)(\mathbf{x}_\nu - \langle x_\nu \rangle) | \psi \rangle = \eta_{\mu\rho} \mathcal{X}'_\nu{}^\rho \\ \mathcal{P}_{\mu\nu} = \langle \psi | (\mathbf{p}_\mu - \langle p_\mu \rangle)(\mathbf{p}_\nu - \langle p_\nu \rangle) | \psi \rangle = \eta_{\mu\rho} \mathcal{P}'_\nu{}^\rho \\ \varrho_{\mu\nu}^\times = \langle \psi | (\mathbf{p}_\mu - \langle p_\mu \rangle)(\mathbf{x}_\nu - \langle x_\nu \rangle) | \psi \rangle \\ \varrho_{\mu\nu}^\times = \langle \psi | (\mathbf{x}_\nu - \langle x_\nu \rangle)(\mathbf{p}_\mu - \langle p_\mu \rangle) | \psi \rangle \\ \varrho_{\mu\nu} = \frac{1}{2}(\varrho_{\mu\nu}^\times + \varrho_{\nu\mu}^\times) \end{cases} \quad (85)$$

From the commutation relation $[\mathbf{p}_\mu, \mathbf{x}_\nu] = i\eta_{\mu\nu}$ we obtain the relations

$$\varrho_{\mu\nu}^\times - \varrho_{\nu\mu}^\times = i\eta_{\mu\nu} \Rightarrow \begin{cases} \varrho_{\mu\nu}^\times = \varrho_{\nu\mu}^\times + i\eta_{\mu\nu} = \varrho_{\mu\nu} + \frac{i}{2}\eta_{\mu\nu} \\ \varrho_{\mu\nu}^\times = \varrho_{\nu\mu}^\times - i\eta_{\mu\nu} = \varrho_{\mu\nu} - \frac{i}{2}\eta_{\mu\nu} \\ \varrho_{\mu\nu} = \frac{1}{2}(\varrho_{\mu\nu}^\times + \varrho_{\nu\mu}^\times) = \varrho_{\nu\mu}^\times - \frac{i}{2}\eta_{\mu\nu} = \varrho_{\mu\nu}^\times + \frac{i}{2}\eta_{\mu\nu} \end{cases} \quad (86)$$

Like in the references [11, 12], the LCT (83) is considered to be a change of observational frame of reference. It is then the momenta and coordinates operators which change but not the state $|\psi\rangle$.

Under the action of the LCT (83), the mean values and statistical variance-covariance become

$$\begin{cases} \langle x'_\mu \rangle = \langle \psi | x'_\mu | \psi \rangle = \eta_{\mu\nu} \langle \psi | x^{\nu'} | \psi \rangle = \eta_{\mu\nu} \langle x^{\nu'} \rangle \\ \langle p'_\mu \rangle = \langle \psi | p'_\mu | \psi \rangle = \eta_{\mu\nu} \langle \psi | p^{\nu'} | \psi \rangle = \eta_{\mu\nu} \langle p^{\nu'} \rangle \\ \mathcal{X}'_{\mu\nu} = \langle \psi | (x'_\mu - \langle x'_\mu \rangle)(x'_\nu - \langle x'_\nu \rangle) | \psi \rangle = \eta_{\mu\rho} \mathcal{X}'_{\nu\rho} \\ \mathcal{P}'_{\mu\nu} = \langle \psi | (p'_\mu - \langle p'_\mu \rangle)(p'_\nu - \langle p'_\nu \rangle) | \psi \rangle = \eta_{\mu\rho} \mathcal{P}'_{\nu\rho} \\ \varrho^{\times'}_{\mu\nu} = \langle \psi | (p'_\mu - \langle p'_\mu \rangle)(x'_\nu - \langle x'_\nu \rangle) | \psi \rangle \\ \varrho^{\times'}_{\nu\mu} = \langle \psi | (x'_\nu - \langle x'_\nu \rangle)(p'_\mu - \langle p'_\mu \rangle) | \psi \rangle \\ \varrho'_{\mu\nu} = \frac{1}{2}(\varrho^{\times'}_{\mu\nu} + \varrho^{\times'}_{\nu\mu}) \end{cases} \quad (87)$$

Let us denote respectively $\langle x \rangle$, $\langle p \rangle$, \mathcal{P} , \mathcal{X} and ϱ the $1 \times N$ and $N \times N$ matrices corresponding respectively to the mean values and statistical variance-covariances defined in the relations (85). From the relations (83), (85)–(87) we deduce the law of transformations

$$(\langle p' \rangle \ \langle x' \rangle) = (\langle p \rangle \ \langle x \rangle) \begin{pmatrix} \mathbf{a} & \mathbf{c} \\ \mathbf{b} & \mathbf{d} \end{pmatrix} \quad (88)$$

$$\begin{pmatrix} \mathcal{P}' & \varrho' \\ \varrho'^T & \mathcal{X}' \end{pmatrix} = \begin{pmatrix} \mathbf{a} & \mathbf{c} \\ \mathbf{b} & \mathbf{d} \end{pmatrix}^T \begin{pmatrix} \mathcal{P} & \varrho \\ \varrho^T & \mathcal{X} \end{pmatrix} \begin{pmatrix} \mathbf{a} & \mathbf{c} \\ \mathbf{b} & \mathbf{d} \end{pmatrix} \quad (89)$$

As generalization of the relation (41), and using the relations (83), (88) and (89), we deduce that the general multidimensional LCT-invariant quadratic operator is

$$\Xi^+ = \frac{1}{8}((\mathbf{p} - \langle p \rangle) \ (\mathbf{x} - \langle x \rangle)) \begin{pmatrix} \mathcal{P} & \varrho \\ \varrho^T & \mathcal{X} \end{pmatrix}^{-1} ((\mathbf{p} - \langle p \rangle) \ (\mathbf{x} - \langle x \rangle))^T \quad (90)$$

The lowest eigenvalue of Ξ^+ is equal to $\frac{D}{4}$ and the corresponding eigenstate, denoted $|\langle z \rangle\rangle = |\{\langle z_\mu \rangle\}\rangle$, is the state corresponding to the coordinate wavefunction of the form [11]

$$\langle \{x^\mu\} | \{\langle z_\mu \rangle\} \rangle = \langle x | \langle z \rangle \rangle = e^{iK} \frac{e^{-\mathcal{B}_{\mu\nu}(x^\mu - \langle x^\mu \rangle)(x^\nu - \langle x^\nu \rangle) - i\langle p_\mu \rangle x^\mu}}{[(2\pi)^D (\det \mathcal{X})]^{1/4}} \quad (91)$$

in which e^{iK} is a unitary complex number that does not depend on x^μ . The statistical parameters in the wavefunction (91) corresponds to the state $|\langle z \rangle\rangle$ itself. $\mathcal{B}_{\mu\nu}$ are parameters, related to the momentum and coordinates statistical variance-covariances, which are given by the expression

$$\mathcal{B}_{\mu\nu} = \frac{1}{4}(\eta_{\mu\rho} + 2i\langle \varrho_{\mu\rho} \rangle) \tilde{\mathcal{X}}_\nu^\rho \quad (92)$$

in which $\tilde{\mathcal{X}}_\mu^\rho$ are related to $\mathcal{X}_{\rho\nu}$ by the relation $\tilde{\mathcal{X}}_\mu^\rho \mathcal{X}_{\rho\nu} = \eta_{\mu\nu}$. We have between $\mathcal{P}_{\mu\nu}$, $\varrho_{\mu\nu}$ and $\tilde{\mathcal{X}}_{\mu\nu}$ the relation

$$\mathcal{P}_{\mu\nu} = \frac{1}{4} \tilde{\mathcal{X}}_{\mu\nu} + \varrho_{\mu\alpha} \tilde{\mathcal{X}}^{\alpha\beta} \varrho_{\nu\beta} \quad (93)$$

The expression of LCT-transform of the wavefunction (91) can also be put in the form [11]

$$\langle \{x'^\mu\} | \{\langle z_\mu \rangle\} \rangle = \langle x' | \langle z \rangle \rangle = e^{iK'} \frac{e^{-\mathcal{B}'_{\mu\nu}(x'^\mu - \langle x'^\mu \rangle)(x'^\nu - \langle x'^\nu \rangle) - i\langle p'_\mu \rangle x'^\mu}}{[(2\pi)^D (\det \mathcal{X}')]^{1/4}} \quad (94)$$

From (91) and (94), it can be shown that the state $|\langle z \rangle\rangle$ is a common eigenstate of the operators $\mathbf{z}_\mu = \mathbf{p}_\mu + 2i\mathcal{B}_{\mu\nu}\mathbf{x}^\nu$ and their LCT-transforms $\mathbf{z}'_\mu = \mathbf{p}'_\mu + 2i\mathcal{B}'_{\mu\nu}\mathbf{x}'^\nu$ [11]

$$\begin{cases} \mathbf{z}_\mu |\langle z \rangle\rangle = (\langle p_\mu \rangle + 2i\mathcal{B}_{\mu\nu} \langle x^\nu \rangle) |\langle z \rangle\rangle = \langle z_\mu \rangle |\langle z \rangle\rangle \\ \mathbf{z}'_\mu |\langle z \rangle\rangle = (\langle p'_\mu \rangle + 2i\mathcal{B}'_{\mu\nu} \langle x'^\nu \rangle) |\langle z \rangle\rangle = \langle z'_\mu \rangle |\langle z \rangle\rangle \end{cases} \quad \forall \ \mu \quad (95)$$

Now, as generalization of the relation (53), it can be shown [11] that the $2D \times 2D$ statistical variance-covariances matrix $\begin{pmatrix} \mathcal{P} & \varrho \\ \varrho^T & \mathcal{X} \end{pmatrix}$ corresponding to the state $|\langle z \rangle\rangle$ can be factorized in the form

$$\begin{pmatrix} \mathcal{P} & \varrho \\ \varrho^T & \mathcal{X} \end{pmatrix} = \begin{pmatrix} \ell & 0 \\ 2ac\ell & a \end{pmatrix}^T \begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix} \begin{pmatrix} \ell & 0 \\ 2ac\ell & a \end{pmatrix} \tag{96}$$

in which a , ℓ and c are $D \times D$ matrices satisfying the following properties

$$\begin{cases} a\ell = \ell a = \frac{1}{2}I_D \text{ (} I_D \text{ being here the } D \times D \text{ identity matrix)} \\ a^T = \eta a \eta \quad a^\dagger = \eta a^T = a \eta \quad \Leftrightarrow a_\mu^{\nu*} = a^{\mu\nu} \\ \ell^T = \eta \ell \eta \quad \ell^\dagger = \eta \ell = \ell^T \eta \Leftrightarrow \ell_\mu^{\nu*} = \ell^{\nu\mu} \\ c^T = 2\eta ac\ell \eta \end{cases} \tag{97}$$

We have also the properties (generalization of the relation (54))

$$\begin{pmatrix} \ell & 0 \\ 2ac\ell & a \end{pmatrix}^{-1} = 2 \begin{pmatrix} a & 0 \\ -c & \ell \end{pmatrix} \tag{98}$$

$$\begin{pmatrix} \mathcal{P} & \varrho \\ \varrho^T & \mathcal{X} \end{pmatrix}^{-1} = 4 \begin{pmatrix} a & 0 \\ -c & \ell \end{pmatrix} \begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix} \begin{pmatrix} a & 0 \\ -c & \ell \end{pmatrix}^T = 4 \begin{pmatrix} \eta \mathcal{X} \eta & -\eta \varrho^T \eta \\ -\eta \varrho \eta & \eta \mathcal{P} \eta \end{pmatrix} \tag{99}$$

Then the expression (90) of the LCT-invariant quadratic operator becomes

$$\Xi^+ = \frac{1}{2} ((\mathbf{p} - \langle p \rangle) \quad (\mathbf{x} - \langle x \rangle)) \begin{pmatrix} a & 0 \\ -c & \ell \end{pmatrix} \begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix} \begin{pmatrix} a & 0 \\ -c & \ell \end{pmatrix}^T \begin{pmatrix} (\mathbf{p} - \langle p \rangle)^T \\ (\mathbf{x} - \langle x \rangle)^T \end{pmatrix} \tag{100}$$

If the momentum and coordinates reduced operators is introduced via the matrix relation

$$(\mathbf{P} \quad \mathbf{x}) = \sqrt{2} (\mathbf{p} - \langle p \rangle \quad \mathbf{x} - \langle x \rangle) \begin{pmatrix} a & 0 \\ -c & \ell \end{pmatrix} \tag{101}$$

the LCT-invariant quadratic operator (100) becomes

$$\Xi^+ = \frac{1}{4} (\mathbf{P} \quad \mathbf{x}) \begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix} \begin{pmatrix} \mathbf{P}^T \\ \mathbf{x}^T \end{pmatrix} = \frac{1}{4} \eta^{\mu\nu} (\mathbf{P}_\mu \mathbf{P}_\nu + \mathbf{x}_\mu \mathbf{x}_\nu) \tag{102}$$

The following operators can be identified to be the ladder operators

$$\begin{cases} \mathbf{z}_\mu = \frac{1}{\sqrt{2}} (\mathbf{P}_\mu + i\mathbf{x}_\mu) = a_\mu^\nu (z_\nu - \langle z_\nu \rangle) \\ \mathbf{z}_\mu^\star = \frac{1}{\sqrt{2}} (\mathbf{P}_\mu - i\mathbf{x}_\mu) = a_\mu^\nu (z_\nu^\dagger - \langle z_\nu^\dagger \rangle) = \mathbf{z}^{\mu\dagger} \\ \mathbf{z}_\mu^\dagger = \frac{1}{\sqrt{2}} (\mathbf{P}_\mu^\dagger - i\mathbf{x}_\mu^\dagger) = a_\mu^{\nu*} (z_\nu^\dagger - \langle z_\nu^\dagger \rangle) = \mathbf{z}^{\mu\star} \end{cases} \tag{103}$$

From the relations in (97), (102), (103) and the CCRs (82), we deduce the relations

$$\begin{cases} \Xi^+ = \frac{1}{4} \delta^{\mu\nu} (\mathbf{z}_\mu \mathbf{z}_\nu^\dagger + \mathbf{z}_\nu^\dagger \mathbf{z}_\mu) = \frac{1}{4} \delta^{\mu\nu} (2\mathbf{z}_\mu \mathbf{z}_\nu^\dagger + D) \\ [\mathbf{z}_\mu, \mathbf{z}_\nu^\dagger] = \delta_{\mu\nu} \quad [\Xi^+, \mathbf{z}_\mu] = -\frac{1}{4} \mathbf{z}_\mu \quad [\Xi^+, \mathbf{z}_\mu^\dagger] = \frac{1}{4} \mathbf{z}_\mu^\dagger \end{cases} \tag{104}$$

with

$$\delta^{\mu\nu} = \delta_{\mu\nu} = \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{if } \mu \neq \nu \end{cases} \tag{105}$$

The relations in (104) show explicitly that \mathbf{z}_μ and \mathbf{z}_μ^\dagger are as expected the LCT-covariant ladder operators associated to the LCT-invariant quadratic operator Ξ^+ . A most general eigenstate of Ξ^+ , denoted $| \{n_\mu\}, \langle z \rangle \rangle = |n, \langle z \rangle \rangle$ is obtained by the relation (n is the set of nonnegative integers n_0, n_1, \dots, n_{D-1})

$$|n, \langle z \rangle \rangle = \left[\prod_{\mu=0}^{D-1} \frac{(\mathbf{z}_\mu^\dagger)^{n_\mu}}{\sqrt{n_\mu!}} \right] | \langle z \rangle \rangle \tag{106}$$

The corresponding eigenvalue equation is

$$\Xi^+ |n, \langle z \rangle \rangle = \frac{1}{4} \left[\sum_{\mu=0}^{D-1} (2n_\mu + 1) \right] |n, \langle z \rangle \rangle = \left[\frac{1}{2} \left(\sum_{\mu=0}^{D-1} n_\mu \right) + \frac{D}{4} \right] |n, \langle z \rangle \rangle \tag{107}$$

As generalization of the operator introduced in the relation (69), we may consider the following operators

$$\begin{cases} \aleph_{\mu\nu} = \frac{1}{2}(\mathfrak{z}_\mu^\dagger \mathfrak{z}_\nu + \mathfrak{z}_\nu \mathfrak{z}_\mu^\dagger) \\ \aleph = \delta^{\mu\nu} \aleph_{\mu\nu} = \delta^{\mu\nu} \mathfrak{z}_\mu^\dagger \mathfrak{z}_\nu = 2\Xi^+ - \frac{D}{2} \end{cases} \quad (108)$$

The operator \aleph , which is a generalization of the operator in (69), is also an LCT-invariant quadratic operator that we may call bosonic LCT-invariant quadratic operator.

5.2. Spin representation of LCTs and associated invariant quadratic operators

The law of transformations of the $1 \times D$ matrices \mathfrak{p} , \mathfrak{x} and \mathfrak{z} which corresponds respectively to the reduced operators \mathfrak{p}_μ , \mathfrak{x}_μ and lowering operator \mathfrak{z}_μ can be written in the form [11]

$$\begin{cases} \mathfrak{p}'_\mu = \Pi'_\mu \mathfrak{p}_\nu + \Theta'_\mu \mathfrak{x}_\nu \\ \mathfrak{x}'_\mu = -\Theta'_\mu \mathfrak{p}_\nu + \Pi'_\mu \mathfrak{x}_\nu \end{cases} \Leftrightarrow (\mathfrak{p}' \ \mathfrak{x}') = (\mathfrak{p} \ \mathfrak{x}) \begin{pmatrix} \Pi & -\Theta \\ \Theta & \Pi \end{pmatrix} \quad (109)$$

$$\mathfrak{z}'_\mu = (\Pi'_\mu - i\Theta'_\mu) \mathfrak{z}_\mu \Leftrightarrow \mathfrak{z}' = \mathfrak{z}(\Pi - i\Theta) = \mathfrak{z}\Omega \quad (110)$$

with the $2D \times 2D$ matrix $\begin{pmatrix} \Pi & -\Theta \\ \Theta & \Pi \end{pmatrix}$ belonging to the group intersection [11]

$$\mathbb{G} \cong Sp(2D_+, 2D_-) \cap O(2D_+, 2N_-) \cong Sp(2D_+, 2D_-) \cap SO_0(2D_+, 2D_-) \quad (111)$$

in which $SO_0(2D_+, 2D_-)$ is the identity component of indefinite special orthogonal group $SO(2D_+, 2D_-)$. And the matrix $\Omega = \Pi - i\Theta$ belongs to the pseudo-unitary group $U(D_+, D_-)$. One has the group isomorphism

$$\mathbb{G} \cong Sp(2D_+, 2D_-) \cap SO_0(2D_+, 2D_-) \cong U(D_+, D_-) \quad (112)$$

The topological double cover of \mathbb{G} is a subgroup \mathbb{S} of the spin group $Spin(2D_+, 2D_-)$ which is the double cover of $SO_0(2D_+, 2D_-)$. The relation between \mathbb{S} and \mathbb{G} can be defined with a surjective covering map \mathscr{U} according to the relation

$$\begin{cases} \mathscr{U}: \mathbb{S} \rightarrow \mathbb{G} \\ \mathbb{S} \rightarrow \mathfrak{g} = \begin{pmatrix} \Pi & -\Theta \\ \Theta & \Pi \end{pmatrix} \Leftrightarrow \begin{cases} (\mathfrak{p}' \ \mathfrak{x}') = (\mathfrak{p} \ \mathfrak{x}) \begin{pmatrix} \Pi & -\Theta \\ \Theta & \Pi \end{pmatrix} \\ \mathfrak{z}' = \mathfrak{z}\mathscr{S}^{-1} \end{cases} \end{cases} \quad (113)$$

with \mathfrak{z} the operator

$$\mathfrak{z} = \frac{1}{\sqrt{2}}(\alpha^\mu \mathfrak{p}_\mu + \beta^\mu \mathfrak{x}_\mu) \quad (114)$$

in which α^μ and β^μ are the generators of the Clifford algebra $\mathscr{C}\ell(2D_+, 2D_-)$. They verify the following anticommutation relations

$$\begin{cases} \alpha^\mu \alpha^\nu + \alpha^\nu \alpha^\mu = 2\eta^{\mu\nu} \\ \beta^\mu \beta^\nu + \beta^\nu \beta^\mu = 2\eta^{\mu\nu} \\ \alpha^\mu \beta^\nu + \beta^\nu \alpha^\mu = 0 \end{cases} \quad (115)$$

As we have $(\alpha^\mu)^2 = (\beta^\mu)^2 = \eta^{\mu\mu} = \pm 1$, we may add the following properties

$$\begin{cases} \alpha^{\mu\dagger} = \alpha^\mu \text{ and } \beta^{\mu\dagger} = \beta^\mu & \text{if } \eta^{\mu\mu} = 1 \\ \alpha^{\mu\dagger} = -\alpha^\mu \text{ and } \beta^{\mu\dagger} = -\beta^\mu & \text{if } \eta^{\mu\mu} = -1 \end{cases} \quad (116)$$

The relation (113) defines a spin representation of LCTs, the element \mathscr{S} of the group \mathbb{S} corresponding to this spin representation can be put in the form

$$\mathscr{S} = e^{\mathfrak{s}} \quad (117)$$

with \mathfrak{s} an element of the Lie algebra \mathfrak{s} of the Lie group \mathbb{S} :

$$\mathfrak{s} = \mathfrak{s}_{\mu\nu} \Xi^{\mu\nu} \quad (118)$$

in which the set $\{\Xi^{\mu\nu}\}$ is a basis of the Lie algebra \mathfrak{s} and we have

$$\Xi^{\mu\nu} = \begin{cases} \frac{1}{4}[(\alpha^\mu \alpha^\nu + \beta^\mu \beta^\nu) + i(\alpha^\mu \beta^\nu + \alpha^\nu \beta^\mu)] & \text{for } \mu \neq \nu \\ \frac{1}{2}i\alpha^\mu \beta^\mu & \text{for } \mu = \nu \end{cases} \quad (119)$$

The number of elements of the family $\{\Xi^{\mu\nu}\}$ is equal to D^2 : it is the dimension of \mathfrak{s} as vectorial space. If we introduce the operators ζ^μ , $\zeta^{\mu\star}$ and $\zeta^{\mu\dagger}$ through the relations

$$\begin{cases} \zeta^\mu = \frac{1}{2}(\alpha^\mu + i\beta^\mu) \\ \zeta^{\mu\star} = \frac{1}{2}(\alpha^\mu - i\beta^\mu) = \zeta_\mu^\dagger \\ \zeta^{\mu\dagger} = \frac{1}{2}(\alpha^{\mu\dagger} - i\beta^{\mu\dagger}) = \zeta_\mu^\star \end{cases} \quad (120)$$

the expressions of the $\Xi^{\mu\nu}$ can be written in a simpler form

$$\Xi^{\mu\nu} = \frac{1}{2}(\zeta^{\mu\star}\zeta^\nu - \zeta^\nu\zeta^{\mu\star}) \quad (121)$$

The anticommutation relations corresponding to the operators ζ^μ and $\zeta^{\mu\dagger} = \zeta_\mu^\star$ can be deduced from the relations (115) and (120), we obtain

$$\begin{cases} \zeta^\mu\zeta^\nu + \zeta^\nu\zeta^\mu = 0 \\ \zeta^{\mu\star}\zeta^{\nu\star} + \zeta^{\nu\star}\zeta^{\mu\star} = 0 \\ \zeta^\mu\zeta^{\nu\star} + \zeta^{\nu\star}\zeta^\mu = \eta^{\mu\nu} \end{cases} \Leftrightarrow \begin{cases} \zeta^\mu\zeta^\nu + \zeta^\nu\zeta^\mu = 0 \\ \zeta^{\mu\dagger}\zeta^{\nu\dagger} + \zeta^{\nu\dagger}\zeta^{\mu\dagger} = 0 \\ \zeta^\mu\zeta^{\nu\dagger} + \zeta^{\nu\dagger}\zeta^\mu = \delta^{\mu\nu} \end{cases} \quad (122)$$

According to the relation (122), the operators ζ^μ and $\zeta^{\mu\dagger} = \zeta_\mu^\star$ have the properties of fermionic ladder operators (while the operators \mathfrak{z}_μ and \mathfrak{z}_μ^\dagger in (104) have the properties of bosonic ladder operators). We may introduce a quadratic operator Σ defined by the following relation μ

$$\begin{cases} \Sigma^{\mu\nu} = \zeta^{\mu\dagger}\zeta^\nu \\ \Sigma = \delta_{\mu\nu}\Sigma^{\mu\nu} = \delta_{\mu\nu}\zeta^{\mu\dagger}\zeta^\nu \end{cases} \quad (123)$$

we have the commutation relations

$$\begin{cases} [\Sigma, \zeta^\mu] = -\zeta^\mu \\ [\Sigma, \zeta^{\mu\dagger}] = \zeta^{\mu\dagger} \end{cases} \quad (124)$$

Using the relations (121)–(123), it can be checked that Σ commutes with the generators $\Xi^{\mu\nu}$ of the Lie algebra. It follows from (117) and (118) that Σ commutes with any element \mathcal{S} of the group \mathbb{S} which corresponds to the spin representation of LCTs. In other words, Σ is an LCT-invariant quadratic operator. Given the relations (122)–(124), we may call it the fermionic LCT-invariant quadratic operator (like we have called the operator \aleph in (108) a bosonic LCT-invariant quadratic operator).

Now, using the relations (103), (104), (114), (120) and (122), it can be deduced that we have the relation

$$(\mathfrak{z})^2 = (\delta^{\mu\nu}\mathfrak{z}_\mu^\dagger\mathfrak{z}_\nu + \delta_{\mu\nu}\zeta^{\mu\dagger}\zeta^\nu) = \aleph + \Sigma \quad (125)$$

in which \aleph is the bosonic LCT-invariant quadratic operator in (111) and Σ the fermionic LCT-invariant quadratic operator in (123). The operator $(\mathfrak{z})^2$ itself is then an LCT-invariant quadratic operator that we may call the mixed (bosonic-fermionic) LCT-invariant quadratic operator.

5.3. Eigenstates of the invariant quadratic operators

Let us denote

$$|n, \mathcal{f}, \langle z \rangle\rangle = |\{n_\mu\}, \{\mathcal{f}^\mu\}, \{\langle z_\mu \rangle\}\rangle \quad (126)$$

the common eigenstates of the number-like operators $\aleph_{\mu\mu} = \mathfrak{z}_\mu^\dagger\mathfrak{z}_\mu$ and $\Sigma^{\mu\mu} = \zeta^{\mu\dagger}\zeta^\mu$ with the eigenvalue equations

$$\begin{cases} \aleph_{\mu\mu}|n, \mathcal{f}, \langle z \rangle\rangle = n_\mu|n, \mathcal{f}, \langle z \rangle\rangle \\ \Sigma^{\mu\mu}|n, \mathcal{f}, \langle z \rangle\rangle = f^\mu|n, \mathcal{f}, \langle z \rangle\rangle \end{cases} \quad (127)$$

and which satisfy the relation

$$\langle n, \mathcal{f}, \langle z \rangle | \mathfrak{z}_\mu | n, \mathcal{f}, \langle z \rangle \rangle = \langle z_\mu \rangle \quad (128)$$

In the relation (126) n refers to the set of the parameters n_μ , \mathcal{f} to the set of the parameters \mathcal{f}^μ and $\langle z \rangle$ to the set of the parameters z_μ ($\mu = 0, 1, \dots, D - 1$) with D the dimension of the pseudo-Euclidian space that is considered. We may introduce the following quantities

$$\begin{cases} |n\rangle = n_0 + n_1 + \dots n_{D-1} \\ |\mathcal{A}\rangle = \mathcal{A}^0 + \mathcal{A}^1 \dots + \mathcal{A}^{D-1} \end{cases} \quad (129)$$

with this relation (129), the eigenvalue equations of the LCT-invariant quadratic operators \aleph , Σ and $(\mathcal{A})^2$ implicated in the relation (125) can be written in the compact forms

$$\begin{cases} \aleph|n, \mathcal{A}, \langle z\rangle\rangle = |n||n, \mathcal{A}, \langle z\rangle\rangle \\ \Sigma|n, \mathcal{A}, \langle z\rangle\rangle = |\mathcal{A}|n, \mathcal{A}, \langle z\rangle\rangle \\ (\mathcal{A})^2|n, \mathcal{A}, \langle z\rangle\rangle = (|n| + |\mathcal{A}|)|n, \mathcal{A}, \langle z\rangle\rangle \end{cases} \quad (130)$$

The degeneracy $g_{|n|}$ of $|n|$ and $g_{|\mathcal{A}|}$ of $|\mathcal{A}|$ are respectively:

$$\begin{cases} g_{|n|} = \frac{(|n| + D - 1)!}{|n|!(D - 1)!} \\ g_{|\mathcal{A}|} = \frac{D!}{|\mathcal{A}|!(D - |\mathcal{A}|)!} \end{cases} \quad (131)$$

The states $|0, \mathcal{A}, \langle z\rangle\rangle = |\mathcal{A}, \langle z\rangle\rangle(|n| = 0)$ are also eigenstates of the operators z_μ

$$z_\mu|\mathcal{A}, \langle z\rangle\rangle = \langle z_\mu\rangle|\mathcal{A}, \langle z\rangle\rangle \quad (132)$$

Any state $|n, \mathcal{A}, \langle z\rangle\rangle$ can be deduced from the state $|0, 0, \langle z\rangle\rangle$ ($|n| = 0$ and $|\mathcal{A}| = 0$) or from any other state using the ladder operators $z_\mu, z_\mu^\dagger, \zeta^\mu$ and ζ^μ .

5.4. Example of application in Particle Physics

The idea of using LCT spin representation to obtain a classification of elementary fermions has been developed in the references [11, 12]. The background space that was considered is a pentadimensional pseudo-Euclidian space with signature (1, 4). Based on the results described in these references, we can establish that a basic quantum state of an elementary fermions can be described with a state $|n, \mathcal{A}, \langle z\rangle\rangle$. As in [11, 12], let us consider the following operators:

$$\mathcal{Y}^0 = \frac{1}{2}i\alpha^0\beta^0 \quad \mathcal{Y}^1 = \frac{1}{3}i\alpha^1\beta^1 \quad \mathcal{Y}^2 = \frac{1}{3}i\alpha^2\beta^2 \quad \mathcal{Y}^3 = \frac{1}{3}i\alpha^3\beta^3 \quad \mathcal{Y}^4 = \frac{1}{2}i\alpha^4\beta^4 \quad (133)$$

in which the α^μ and β^μ are the generators of the Clifford algebra $\mathcal{C}\ell(2,8)$. They verify anticommutation relations similar to (115). As shown in [10, 11], the operators I_3 , Y_W and Q corresponding respectively to the weak isospin, weak hypercharge and electric charge of an elementary fermion of the Standard Model can be defined from the operators in (133) by the relations

$$\begin{cases} I_3 = \frac{1}{2}\mathcal{Y}^0 - \frac{1}{2}\mathcal{Y}^4 & Y_W = \mathcal{Y}^0 + \mathcal{Y}^1 + \mathcal{Y}^2 + \mathcal{Y}^3 + \mathcal{Y}^4 \\ Q = \mathcal{Y}^0 + \frac{1}{2}\mathcal{Y}^1 + \frac{1}{2}\mathcal{Y}^2 + \frac{1}{2}\mathcal{Y}^3 = I_3 + \frac{Y_W}{2} \end{cases} \quad (134)$$

From the relations (120), we can deduce the relations

$$\begin{cases} I_3 = \frac{1}{2}(\Sigma^{00} + \Sigma^{44}) - \frac{1}{2} \\ Y_W = \Sigma^{00} - \frac{2}{3}(\Sigma^{11} + \Sigma^{22} + \Sigma^{33}) - \Sigma^{44} + 1 \\ Q = \Sigma^{00} - \frac{1}{3}(\Sigma^{11} + \Sigma^{22} + \Sigma^{33}) = I_3 + \frac{Y_W}{2} \end{cases} \quad (135)$$

It follows from the relation (127) and (135) that the eigenstates of I_3 , Y_W and Q are the states $|n, \mathcal{A}, \langle z\rangle\rangle$. The table 1 below gives a classification of the states $|n, \mathcal{A}, \langle z\rangle\rangle$ which corresponds to a classification of a family of elementary fermions, for a fixed value of n , and according to the values of the eigenvalues $\mathcal{A}^0, \mathcal{A}^1, \mathcal{A}^2, \mathcal{A}^3$ and \mathcal{A}^4 of the operators $\Sigma^{\mu\mu}$ and the eigenvalues $|\mathcal{A}|, I_3, Y_W$ and Q of the operators Σ, I_3, Y_W and Q . It follows from (127) and (135) that

$$\begin{cases} I_3 = \frac{1}{2}(\mathcal{A}^0 + \mathcal{A}^4) - \frac{1}{2} \\ Y_W = \mathcal{A}^0 - \frac{2}{3}(\mathcal{A}^1 + \mathcal{A}^2 + \mathcal{A}^3) - \mathcal{A}^4 + 1 \\ Q = \mathcal{A}^0 - \frac{1}{3}(\mathcal{A}^1 + \mathcal{A}^2 + \mathcal{A}^3) = I_3 + \frac{Y_W}{2} \end{cases} \quad (136)$$

The denomination used for a state $|n, \mathcal{A}, \langle z\rangle\rangle$ in the table 1 corresponds to the denomination of the first generation of elementary fermions of the Standard model of particle physics: ν and $\bar{\nu}$ refer respectively to

Table 1. Classification of the eigenstates of LCT- invariant quadratic operators and correspondance with a classification of elementary fermions

EIGENVALUES OF $\Sigma^{00}, \Sigma^{11}, \Sigma^{22}, \Sigma^{33}, \Sigma^{44}, \Sigma, I_3, Y_W$ AND Q									STATES
f^0	f^1	f^2	f^3	f^4	$ A $	I_3	Y_W	Q	
0	0	0	0	0	0	-1/2	1	0	$ n, \bar{\nu}_L, \langle z \rangle\rangle$
0	0	0	0	1	1	0	0	0	$ n, \nu_R, \langle z \rangle\rangle$
0	0	0	1	0	1	-1/2	1/3	-1/3	$ n, d_L^r, \langle z \rangle\rangle$
0	0	0	1	1	2	0	-2/3	-1/3	$ n, d_R^r, \langle z \rangle\rangle$
0	0	1	0	0	1	-1/2	1/3	-1/3	$ n, d_L^g, \langle z \rangle\rangle$
0	0	1	0	1	2	0	-2/3	-1/3	$ n, d_R^g, \langle z \rangle\rangle$
0	0	1	1	0	2	-1/2	-1/3	-2/3	$ n, \bar{u}_L^b, \langle z \rangle\rangle$
0	0	1	1	1	3	0	-4/3	-2/3	$ n, \bar{u}_R^b, \langle z \rangle\rangle$
0	1	0	0	0	1	-1/2	1/3	-1/3	$ n, d_L^b, \langle z \rangle\rangle$
0	1	0	0	1	2	0	-2/3	-1/3	$ n, d_R^b, \langle z \rangle\rangle$
0	1	0	1	0	2	-1/2	-1/3	-2/3	$ n, \bar{u}_L^g, \langle z \rangle\rangle$
0	1	0	1	1	3	0	-4/3	-2/3	$ n, \bar{u}_R^g, \langle z \rangle\rangle$
0	1	1	0	0	2	-1/2	-1/3	-2/3	$ n, \bar{u}_L^r, \langle z \rangle\rangle$
0	1	1	0	1	3	0	-4/3	-2/3	$ n, \bar{u}_R^r, \langle z \rangle\rangle$
0	1	1	1	0	3	-1/2	-1	-1	$ n, e_L, \langle z \rangle\rangle$
0	1	1	1	1	4	0	-2	-1	$ n, e_R, \langle z \rangle\rangle$
1	0	0	0	0	1	0	2	1	$ n, \bar{e}_R, \langle z \rangle\rangle$
1	0	0	0	1	2	1/2	1	1	$ n, \bar{e}_L, \langle z \rangle\rangle$
1	0	0	1	0	2	0	4/3	2/3	$ n, u_R^r, \langle z \rangle\rangle$
1	0	0	1	1	3	1/2	1/3	2/3	$ n, u_L^r, \langle z \rangle\rangle$
1	0	1	0	0	2	0	4/3	2/3	$ n, u_R^g, \langle z \rangle\rangle$
1	0	1	0	1	3	1/2	1/3	2/3	$ n, u_L^g, \langle z \rangle\rangle$
1	0	1	1	0	3	0	2/3	1/3	$ n, \bar{d}_R^b, \langle z \rangle\rangle$
1	0	1	1	1	4	1/2	-1/3	1/3	$ n, \bar{d}_L^b, \langle z \rangle\rangle$
1	1	0	0	0	2	0	4/3	2/3	$ n, u_R^b, \langle z \rangle\rangle$
1	1	0	0	1	3	1/2	1/3	2/3	$ n, u_L^b, \langle z \rangle\rangle$
1	1	0	1	0	3	0	2/3	1/3	$ n, \bar{d}_R^g, \langle z \rangle\rangle$
1	1	0	1	1	4	1/2	-1/3	1/3	$ n, \bar{d}_L^g, \langle z \rangle\rangle$
1	1	1	0	0	3	0	2/3	1/3	$ n, \bar{d}_R^r, \langle z \rangle\rangle$
1	1	1	0	1	4	1/2	-1/3	1/3	$ n, \bar{d}_L^r, \langle z \rangle\rangle$
1	1	1	1	0	4	0	0	0	$ n, \bar{\nu}_R, \langle z \rangle\rangle$
1	1	1	1	1	5	1/2	-1	0	$ n, \nu_L, \langle z \rangle\rangle$

neutrino and antineutrino, e and \bar{e} refer respectively to negaton and positon, u and \bar{u} refer respectively to up type quark and antiquark, d and \bar{d} refer respectively to down type quark and antiquark. The index L and R correspond to chirality, Left or Right and the exponents r, g and b correspond respectively to strong color charge: red, green and blue. The table 1 correspond to one generation of fermions but it is obtained for a fixed value of n i.e. for a fixed values of the parameters n_0, n_1, n_2, n_3 and n_4 . These parameters may help in the understanding of the existence of multiple generations of fermions. The table 1 suggests also the existence of sterile neutrinos.

6. Main results

Some of the main results established through this work are the followings:

- There is a relation between the quantum theory of linear harmonic oscillator and Linear Canonical Transformations. According to the relations (2)–(4), the Hamiltonian of a linear harmonic oscillator can be considered as an invariant quadratic operator associated to some particular LCTs.
- According to the relations (7) and (8), the angular frequency of a linear harmonic oscillator can be directly related to the statistical variance of the momentum operator and the Hamiltonian operator can be directly related to the momentum dispersion operator.
- According to the section 3, the theory of a linear harmonic oscillator with nonzero mean values of coordinate and momentum can be exploited to obtain a new description of the motion of a non-relativistic free quantum particle which is compatible with the LCT-covariance and solve some inconvenience of the old description.

This new description, which takes into account quantum fluctuation of momentum, can be used to obtain a phase space representation of quantum theory.

- According to the section 4, it is possible to relate quantum statistical parameters like momentum statistical variance, which are present in the expression of LCT-invariant quadratic operators, with thermodynamic variables such as temperature, pressure and volume through the formalism of quantum statistical mechanics. This relation permit to obtain new quantum corrections to the ideal gas state equation. These new quantum corrections may be considered to conceive experimental tests for the validity of the new description of free nonrelativistic quantum particle that is introduced in the section 3.
- According to the relations (31), (41) and (60), a first invariant quadratic operator associated to general monodimensional LCTs, denoted Ξ^+ , can be considered as a generalization of a reduced momentum dispersion operator. Another invariant quadratic operator, denoted \mathcal{N} , which is directly related to Ξ^+ by the relation (69) can be considered as a number operator of bosonic quasiparticles. We may then call \mathcal{N} a bosonic invariant quadratic operator. The multidimensional generalization of Ξ^+ and \mathcal{N} are considered in the relations (90), (100), (102), (104) and (108). The common eigenstates of these operators which are considered in the relation (62), (106) and (107) can be considered as generalization of the basic state of a linear harmonic oscillator and generalization of what are called coherent states and squeezed states in the literature [16–19].
- Following the relation (122)–(125) and (130), a fermionic analog of the bosonic LCT-invariant quadratic operator \mathcal{N} can be also considered. This fermionic LCT-invariant quadratic operator is denoted Σ . The sum $(\mathcal{N})^2 = \mathcal{N} + \Sigma$ considered in the relations (125) and (130) is itself an LCT-invariant quadratic operator. The eigenvalues equation corresponding to these operators are given in (130).
- The common eigenstates of the LCT-invariant quadratic operators associated to a pentadimensional pseudo-Euclidian space with signature (1, 4) correspond to basic quantum states of elementary fermions. Following the table 1, a classification of these basic states lead to a classification of elementary fermions which is compatible with the Standard Model of particle physics, suggest the existence of sterile neutrinos and may help in the understanding of the existence of more than one fermions family.

7. Conclusion

The expressions of the general LCT-invariant quadratic operators are given by the relations (108), (123) and (125). Their eigenvalues equations are given in the relation (130).

The analysis of the results enumerated in the previous section show that the LCT-invariant quadratic operators and their eigenstates can have important roles to play in physics. These results can especially help in the understanding and resolution of some of the main open fundamental problems related to quantum theory, statistical physics, relativistic quantum thermodynamics and particle physics like the question considered in the reference [20], the unsolved questions related to sterile neutrinos [12] and the origin of multiple generations of fermions.

Given the possible link between LCT-covariance and fundamental interactions highlighted in the references [11, 12], the results obtained through this work can also be a particular help in the establishment of a unified theory of interactions which includes gravity.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix A. Harmonic oscillator with nonzero mean values, $\langle p \rangle$ and $\langle x \rangle$, of momentum and coordinate

Instead of an oscillator with the mean values $\langle \psi | \mathbf{p} | \psi \rangle = \langle p \rangle = 0$ and $\langle \psi | \mathbf{x} | \psi \rangle = \langle x \rangle = 0$ considered in the section 2 of the main text, we may consider an oscillator that may have non-null momentum and coordinate

mean values: $\langle p \rangle \neq 0$ and $\langle x \rangle \neq 0$. This kind of oscillator can be used to have a more realistic model of a nonrelativistic quantum free particle as described in the section 3.

As we have $\langle \psi | \mathbf{p} | \psi \rangle \neq 0$ and $\langle \psi | \mathbf{x} | \psi \rangle \neq 0$ i.e. the oscillator has a mean motion (which is a classical-like motion), we should add to the expression of the Hamiltonian (8) of an oscillator with no mean motion a classical-like kinetic energy term $\frac{\langle \psi | \mathbf{p} | \psi \rangle^2}{2m}$ corresponding to the nonzero value of $\langle \psi | \mathbf{p} | \psi \rangle$. So instead of the particular case corresponding to (8), we should have the new (more general) Hamiltonian:

$$\mathbf{H} = \frac{\langle \psi | \mathbf{p} | \psi \rangle^2}{2m} + \frac{\mathfrak{Q}^+}{m} \quad (\text{A1})$$

- The first part of this expression i.e. the quantity

$$\frac{\langle \psi | \mathbf{p} | \psi \rangle^2}{2m} \quad (\text{A2})$$

corresponds to the classical-like kinetic energy associated to the mean motion of the oscillator.

- The second part

$$\frac{\mathfrak{Q}^+}{m} = \frac{1}{2m} [(\mathbf{p} - \langle \psi | \mathbf{p} | \psi \rangle)^2 + 4\mathcal{B}(\mathbf{x} - \langle \psi | \mathbf{x} | \psi \rangle)^2] \quad (\text{A3})$$

which is similar to the expression in the relation (8) corresponds to the quantum oscillation. \mathfrak{Q}^+ being the momentum dispersion operator [10].

The Hamiltonian operator in (A1) depends itself explicitly on the state $|\psi\rangle$ and the solutions $|\psi(t)\rangle$ of the evolution equation

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = \mathbf{H} |\psi(t)\rangle \quad (\text{A4})$$

should be searched as self-consistent solutions.

Firstly, it can be remarked that the momentum dispersion operator \mathfrak{Q}^+ has as eigenstates the states $|n, \langle p \rangle, \langle x \rangle, \mathcal{B}\rangle$ which correspond to the coordinate wavefunctions [10]:

$$\langle x | n, \langle p \rangle, \langle x \rangle, \mathcal{B} \rangle = \left(\frac{2\mathcal{B}}{\pi\hbar^2} \right)^{1/4} \frac{H_n \left(\frac{\sqrt{2\mathcal{B}}}{\hbar} (x - \langle x \rangle) \right)}{\sqrt{2^n n!}} e^{-\frac{\mathcal{B}}{\hbar^2} (x - \langle x \rangle)^2 + \frac{i}{\hbar} \langle p \rangle x} \quad (\text{A5})$$

and the corresponding eigenvalue equation is

$$\mathfrak{Q}^+ |n, \langle p \rangle, \langle x \rangle, \mathcal{B}\rangle = (2n + 1)\mathcal{B} |n, \langle p \rangle, \langle x \rangle, \mathcal{B}\rangle \quad (\text{A6})$$

It can also be verified explicitly that we have the relations [10]

$$\begin{cases} \langle n, \langle p \rangle, \langle x \rangle, \mathcal{B} | \mathbf{p} | n, \langle p \rangle, \langle x \rangle, \mathcal{B} \rangle = \langle p \rangle \\ \langle n, \langle p \rangle, \langle x \rangle, \mathcal{B} | \mathbf{x} | n, \langle p \rangle, \langle x \rangle, \mathcal{B} \rangle = \langle x \rangle \end{cases} \quad (\text{A7})$$

And

$$\begin{cases} \langle n, \langle p \rangle, \langle x \rangle, \mathcal{B} | (\mathbf{p} - \langle p \rangle)^2 | n, \langle p \rangle, \langle x \rangle, \mathcal{B} \rangle = (2n + 1)\mathcal{B} \\ \langle n, \langle p \rangle, \langle x \rangle, \mathcal{B} | (\mathbf{x} - \langle x \rangle)^2 | n, \langle p \rangle, \langle x \rangle, \mathcal{B} \rangle = (2n + 1)\mathcal{A} = (2n + 1) \frac{\hbar^2}{4\mathcal{B}} \end{cases} \quad (\text{A8})$$

The relations (A7) and (A8) justify the notation $|n, \langle p \rangle, \langle x \rangle, \mathcal{B}\rangle$ used for the eigenstates of \mathfrak{Q}^+ corresponding to the wavefunctions (A5). Moreover, these wavefunctions are just generalization of the wavefunctions in the relation (9) in the main text.

Secondly, as the oscillator have a mean motion, we should suppose that the operators \mathbf{p} and \mathbf{x} themselves are time-dependent. It can be established that adequate relations which permit to have self-consistent solutions for the equation (A4) and compatibility with the Ehrenfest Theorem are

$$\begin{cases} \frac{\partial \mathbf{p}}{\partial t} = \frac{\partial \langle p \rangle}{\partial t} = 0 \Rightarrow \frac{\partial}{\partial t} [(\mathbf{p} - \langle p \rangle)] = 0 \\ \frac{\partial \mathbf{x}}{\partial t} = \frac{\partial \langle x \rangle}{\partial t} = \frac{\langle p \rangle}{m} \Rightarrow \frac{\partial}{\partial t} [(\mathbf{x} - \langle x \rangle)] = 0 \end{cases} \quad (\text{A9})$$

Given the relations (A1), (A6)–(A9), it can be verified that the basic self-consistent solutions of the evolution equation (A4) are the states

$$|\psi_n(t)\rangle = e^{-\frac{i}{\hbar}\varepsilon_n t}|n, \langle p\rangle, \langle x\rangle, \mathcal{B}\rangle \quad (\text{A10})$$

in which ε_n is an eigenvalue of the Hamiltonian operator (A1). Its expression, as expected is

$$\varepsilon_n = \frac{\langle\langle\psi_n(t)|\mathbf{p}|\psi_n(t)\rangle\rangle^2}{2m} + (2n + 1)\frac{\mathcal{B}}{m} = \frac{\langle p\rangle^2}{2m} + (2n + 1)\frac{\mathcal{B}}{m} \quad (\text{A11})$$

Explicitly, the eigenvalue equation of the Hamiltonian operator (A1) is then

$$\mathbf{H}|\psi_n(t)\rangle = \left[\frac{\langle p\rangle^2}{2m} + (2n + 1)\frac{\mathcal{B}}{m} \right] |\psi_n(t)\rangle \quad (\text{A12})$$

Using the relation (A5), it can be checked that a state $|0, \langle p\rangle, \langle x\rangle, \mathcal{B}\rangle (n = 0)$ is an eigenstate of the operator $\mathbf{z} = \mathbf{p} + \frac{2i}{\hbar}\mathcal{B}\mathbf{x}$ with the eigenvalue $\langle z\rangle = \langle n, \langle p\rangle, \langle x\rangle, \mathcal{B}|\mathbf{z}|n, \langle p\rangle, \langle x\rangle, \mathcal{B}\rangle = \langle p\rangle + \frac{2i}{\hbar}\mathcal{B}\langle x\rangle$

$$\mathbf{z}|0, \langle p\rangle, \langle x\rangle, \mathcal{B}\rangle = [\langle p\rangle + \frac{2i}{\hbar}\mathcal{B}\langle x\rangle]|0, \langle p\rangle, \langle x\rangle, \mathcal{B}\rangle = \langle z\rangle|0, \langle p\rangle, \langle x\rangle, \mathcal{B}\rangle \quad (\text{A13})$$

It follows that we may use the simpler notations $|\langle z\rangle\rangle = |0, \langle p\rangle, \langle x\rangle, \mathcal{B}\rangle$ and $|n, \langle p\rangle, \langle x\rangle, \mathcal{B}\rangle = |n, \langle z\rangle\rangle$ (as in the [11]). Using these simpler notations, the basic self-consistent solutions (A10) of the evolution equation (A4) can be written as

$$|\psi_n(t)\rangle = e^{-\frac{i}{\hbar}\varepsilon_n t}|n, \langle p\rangle, \langle x\rangle, \mathcal{B}\rangle = e^{-\frac{i}{\hbar}\varepsilon_n t}|n, \langle z\rangle\rangle \quad (\text{A14})$$

Both a time-dependent state $\psi_n(t)$ and a time independent state n, z are eigenstates of the hamiltonian operator in (A1) with the same eigenvalue ε_n .

As a generalization of the ladder operators \mathbf{z} and \mathbf{z}^\dagger in the relations (11) and (13), we have

$$\begin{cases} \mathbf{z} = \frac{i(z - \langle z\rangle)}{2\sqrt{\mathcal{B}}} = \frac{i}{2\sqrt{\mathcal{B}}}[(\mathbf{p} - \langle p\rangle) - \frac{2i}{\hbar}\mathcal{B}(\mathbf{x} - \langle x\rangle)] \\ \mathbf{z}^\dagger = \frac{i(z - \langle z\rangle)}{2\sqrt{\mathcal{B}}} = \frac{i}{2\sqrt{\mathcal{B}}}[(\mathbf{p} - \langle p\rangle) - \frac{2i}{\hbar}\mathcal{B}(\mathbf{x} - \langle x\rangle)] \end{cases} \quad [\mathbf{z}, \mathbf{z}^\dagger] = 1 \quad (\text{A15})$$

$$\begin{cases} \mathbf{z}|n, \langle z\rangle\rangle = \sqrt{n}|n - 1, \langle z\rangle\rangle \\ \mathbf{z}^\dagger|n, \langle z\rangle\rangle = \sqrt{n + 1}|n + 1, \langle z\rangle\rangle \end{cases} \quad (\text{A16})$$

and we have also as generalization of (12)

$$\mathbf{H} = \frac{\langle p\rangle^2}{2m} + \frac{\mathfrak{J}^+}{m} = \frac{\langle p\rangle^2}{2m} + \frac{1}{m}(\mathbf{z}^\dagger\mathbf{z} + \mathbf{z}\mathbf{z}^\dagger) = \frac{\langle p\rangle^2}{2m} + \frac{\mathcal{B}}{m}(2\mathbf{z}^\dagger\mathbf{z} + 1) \quad (\text{A17})$$

It is straightforward to remark that the relations (A15)–(A17) lead also to the eigenvalue equation (A12).

Appendix B. Corrected Partition function of a free non-relativistic particle which takes into account quantum fluctuations of momentum value

We consider the case of a non-relativistic particle in three dimensional space at thermal equilibrium with a bath. It corresponds to the canonical ensemble. We denote respectively P, V and T the pressure, the volume allowed to the particle and the temperature.

The three dimensional generalization of the monodimensional expression of the Hamiltonian operator given in the relation (20) is

$$\mathbf{H} = \frac{\langle \vec{p}\rangle^2}{2m} + \frac{\mathfrak{J}_{11}^+ + \mathfrak{J}_{22}^+ + \mathfrak{J}_{33}^+}{m} = \frac{\langle \vec{p}\rangle^2}{2m} + \frac{1}{m}(2\mathbf{z}_1^\dagger\mathbf{z}_1\mathcal{B}_{11} + 2\mathbf{z}_2^\dagger\mathbf{z}_2\mathcal{B}_{22} + 2\mathbf{z}_3^\dagger\mathbf{z}_3\mathcal{B}_{33} + 3) \quad (\text{B1})$$

For sake of simplicity, we may suppose that we have the particular case $\mathcal{B}_{11} = \mathcal{B}_{22} = \mathcal{B}_{33} = \mathcal{B}$. We have then

$$\mathbf{H} = \frac{\langle \vec{p}\rangle^2}{2m} + \frac{\mathcal{B}}{m}(2\mathbf{z}_1^\dagger\mathbf{z}_1 + 2\mathbf{z}_2^\dagger\mathbf{z}_2 + 2\mathbf{z}_3^\dagger\mathbf{z}_3 + 3) \quad (\text{B2})$$

We suppose that the particle moves inside the volume V around the center of these volume that one can take as the origin on the coordinates systems so we may choose for the position and momentum global mean values $\langle \vec{x}\rangle = \vec{0}$ and $\langle \vec{p}\rangle = \vec{0}$ and we have then $\langle z_1\rangle = 0, \langle z_2\rangle = 0$ and $\langle z_3\rangle = 0$ too. So we finally have for the Hamiltonian

$$\mathbf{H} = \frac{\mathcal{B}}{m}(2\mathbf{z}_1^\dagger \mathbf{z}_1 + 2\mathbf{z}_2^\dagger \mathbf{z}_2 + 2\mathbf{z}_3^\dagger \mathbf{z}_3 + 3) \quad (\text{B3a})$$

The corresponding eigenvalue equation is

$$\mathbf{H}|n, \mathcal{B}\rangle = \left[(2n_1 + 2n_2 + 2n_3 + 3) \frac{\mathcal{B}}{m} \right] |n, \mathcal{B}\rangle \quad (\text{B3b})$$

with $|n, \mathcal{B}\rangle = |n_1, n_2, n_3, \langle z_1 \rangle = 0, \langle z_2 \rangle = 0, \langle z_3 \rangle = 0\rangle$ the eigenstate corresponding to the eigenvalue

$$\varepsilon_{n_1, n_2, n_3} = (2n_1 + 2n_2 + 2n_3 + 3) \frac{\mathcal{B}}{m} \quad (\text{B4})$$

As it is known in quantum statistical mechanics, the Von Newman entropy of a system described by a density operator ρ is given by the relation

$$S = -k \text{Tr}[\rho \ln \rho] \quad (\text{B5})$$

k is the Boltzmann constant and Tr refers to the trace of an operator. At thermal equilibrium, the density operator ρ and the Hamiltonian \mathbf{H} commute, $\rho\mathbf{H} = \mathbf{H}\rho$, and then, for the case of the particle that is considered, they have the same eigenstates $|n, \mathcal{B}\rangle$. It follows that ρ can then be put in the form

$$\rho = \sum_{n_1, n_2, n_3} q_{n_1, n_2, n_3} |n, \mathcal{B}\rangle \langle n, \mathcal{B}| \quad (\text{B6})$$

in which q_{n_1, n_2, n_3} are the eigenvalues of ρ . Using (B6), (B5) can be put in the form

$$S = -k \sum_{n_1, n_2, n_3} q_{n_1, n_2, n_3} \ln(q_{n_1, n_2, n_3}) \quad (\text{B7})$$

Using the maximum entropy principle corresponding to thermal equilibrium, with the constraints ($U = \langle \mathbf{H} \rangle$ being the thermodynamical internal energy of the particle)

$$\begin{cases} \sum_{n_1, n_2, n_3} q_{n_1, n_2, n_3} = 1 \\ U = \langle \mathbf{H} \rangle = \text{Tr}(\rho\mathbf{H}) = \sum_{n_1, n_2, n_3} q_{n_1, n_2, n_3} \varepsilon_{n_1, n_2, n_3} \end{cases} \quad (\text{B8})$$

We can deduce the expression of ρ and its eigenvalues q_{n_1, n_2, n_3} (which correspond to a canonical ensemble)

$$q_{n_1, n_2, n_3} = \frac{e^{-\beta \varepsilon_{n_1, n_2, n_3}}}{\mathcal{Z}} \Leftrightarrow \rho = \frac{e^{-\beta \mathbf{H}}}{\mathcal{Z}} \quad (\text{B9})$$

with $\beta = \frac{1}{kT}$ and \mathcal{Z} the partition function

$$\mathcal{Z} = \text{Tr}(e^{-\beta \mathbf{H}}) = \sum_{n_1, n_2, n_3} q_{n_1, n_2, n_3} e^{-\beta \varepsilon_{n_1, n_2, n_3}} \quad (\text{B10})$$

Using the expression (B4) of $\varepsilon_{n_1, n_2, n_3}$, it can be calculated that one has

$$\mathcal{Z} = \sum_{n_1, n_2, n_3} e^{-\beta[(2n_1+2n_2+2n_3+3)\frac{\mathcal{B}}{m}]} = \frac{e^{-3\beta\frac{\mathcal{B}}{m}}}{(1 - e^{-2\beta\frac{\mathcal{B}}{m}})^3} = \frac{1}{8sh^3(\beta\frac{\mathcal{B}}{m})} \quad (\text{B11})$$

Another way to calculate the partition function \mathcal{Z} utilizes the decomposition of $e^{-\beta \mathbf{H}}$ in the overcomplete frame $\{|\langle \vec{z} \rangle\rangle\} = \{|\langle z_1 \rangle\rangle, |\langle z_2 \rangle\rangle, |\langle z_3 \rangle\rangle\}$ which corresponds to the phase space representation (generalization of the relation (29))

$$e^{-\beta \mathbf{H}} = \int |\langle \vec{z} \rangle\rangle \langle \langle \vec{z} \rangle| e^{-\beta \mathbf{H}} |\langle \vec{z} \rangle'\rangle \langle \langle \vec{z} \rangle'| \frac{d^3 \langle \vec{p} \rangle d^3 \langle \vec{x} \rangle}{h^3} \frac{d^3 \langle \vec{p}' \rangle d^3 \langle \vec{x}' \rangle}{h^3} \quad (\text{B12})$$

The expression of the scalar product $\langle \langle \vec{z} \rangle | n, \mathcal{B} \rangle$ can be used to calculate $\mathcal{Z} = \text{Tr}(e^{-\beta \mathbf{H}})$ from (B12) and this calculation gives the same result as in the relation (B11)

$$\mathcal{Z} = \text{Tr}(e^{-\beta \mathbf{H}}) = \int \langle \langle \vec{z} \rangle | e^{-\beta \mathbf{H}} | \langle \vec{z} \rangle \rangle \frac{d^3 \langle \vec{p} \rangle d^3 \langle \vec{x} \rangle}{h^3} = \frac{1}{8sh^3(\beta\frac{\mathcal{B}}{m})} \quad (\text{B13})$$

The expression (B11) or (B13) gives the corrected partition function of a free nonrelativistic quantum particle which corresponds to the three dimensional generalization of the corrected Hamiltonian (20). These corrected Hamiltonian and partition function take into account the existence of quantum fluctuation of the momentum value.

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